Quasi-Logarithmic Structures

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Abstract

Quasi-logarithmic combinatorial structures are a class of random decomposable combinatorial structures which generalize the logarithmic class studied by Arratia, Barbour and Tavaré (2000, 2003, 2005). This extension is motivated by work of Zhang (1996a, 1996b) who studies additive arithmetic semigroups that satisfy a condition which leads to a counterpart of generalized integers as introduced by Beurling (1937). These semigroups are quasi-logarithmic but not necessarily logarithmic structures.

Following the ideas of Arratia et al. (2003) in the logarithmic context, quasi-logarithmic structures are introduced in the present work as combinatorial structures whose component spectra $C^{(n)}$, $n \in \mathbb{N}$, satisfy a conditioning relation of the form $\mathscr{L}(C^{(n)}) = \mathscr{L}(Z_1, \ldots, Z_n | \sum_{i=1}^{n} iZ_i = n)$ for all $n \in \mathbb{N}$, where Z_i, $i \in \mathbb{N}$, is a sequence of mutually independent random variables that satisfies certain conditions. Loosely speaking, the Z_i are supposed to be distributionally close to Poisson distributions, and $i\mathbb{E}Z_i$ has to vary around a positive constant as $i \to \infty$ without being equal to 0 too often.

A large part of this work deals with the asymptotic behaviour of the component spectrum of quasi-logarithmic structures. In order to obtain these asymptotic approximations to the component spectrum, it is necessary first to establish an approximation to the scaled sums $n^{-1} \sum_{i=1}^{n} iZ_i$, $n \in \mathbb{N}$, in terms of the Dickman distribution. We obtain convergence rates for the Wasserstein distance between the distribution of the scaled sums and the Dickman distribution, and derive a local limit theorem. The Wasserstein approximation in turn requires an argument that refines the Mineka coupling (Mineka, 1973; Rösler, 1977a; 1977b) by incorporating a blocking construction, leading to exponentially sharper coupling rates for the sums in question.

Applications include analogues for quasi-logarithmic structures of the fundamental lemma and the main theorem of Kubilius (1962, 1964) from probabilistic number theory. The former deals with distributional asymptotics of the spectrum of small components (including convergence rates for the total variation distance relying on our coupling); the latter is a combinatorial version of the bounded variances central limit theorem. We also derive an analogue of the Erdős-Kac central limit theorem (Erdős and Kac, 1940).

Another application deals with quasi-logarithmic additive arithmetic semigroups. We prove that in these semigroups all partition sets have asymptotic density using results of Compton (1989) and an extended results of Woods (1997). In this way, a logical limit theorem for monadic second-order logic is established, which extends previous results of Bell and Burris (2003), Granovsky and Stark (2006) and Stark (2006).

Zusammenfassung

Quasi-logarithmische Strukturen sind zerlegbare kombinatorische Strukturen welche die Klasse der logarithmischen Strukturen, wie sie unter anderem von Arratia, Barbour und Tavaré (2000, 2003, 2005) untersucht wird, verallgemeinern. Die Erweiterung von logarithmischen auf quasi-logarithmische Strukturen umfasst insbesondere additive arithmetische Halbgruppen wie sie von Zhang (1996a, 1996b) eingeführt worden sind; diese erfüllen das Pendant einer Bedingung, die von Beurling (1937) bei verallgemeinerten ganzen Zahlen für die Herleitung von Primzahlsätzen verwendet worden ist.

In dieser Arbeit werden quasi-logarithmische Strukturen analog zu logarithmischen Strukturen in Arratia et al. (2003) eingeführt, nämlich als kombinatorische Strukturen deren Komponentenspektren $C^{(n)}$, $n \in \mathbb{N}$, einer Relation der Form $\mathscr{L}(C^{(n)}) = \mathscr{L}(Z_1, \ldots, Z_n | \sum_{i=1}^{n} iZ_i = n)$, für alle $n \in \mathbb{N}$, genügen. Dabei ist Z_i, $i \in \mathbb{N}$, eine Folge von unabhängigen Zufallsvariablen deren Verteilungen hinreichend nahe an Poisson-Verteilungen liegen, mit der Eigenschaft, dass die $i\mathbb{E}Z_i$ in einem gewissen Sinn um einen positiven konstanten Wert oszillieren, ohne dabei den Wert 0 zu häufig anzunehmen.

Ein Grossteil dieser Arbeit widmet sich dem asymptotischen Verhalten des Komponentenspektrums quasi-logarithmischer Strukturen. Um solche asymptotischen Approximationen zu erhalten, werden zunächst die skalierten Summen $n^{-1} \sum_{i=1}^{n} iZ_i$, $n \in \mathbb{N}$, mit der Dickman-Verteilung approximiert. Wir erhalten insbesondere Konvergenzraten für die Wasserstein-Distanz zwischen der Verteilung der skalierten Summen und der Dickman-Verteilung. Weiter erhalten wir einen lokalen Grenzwertsatz für die Punktwahrscheinlichkeiten von $\sum_{i=1}^{n} iZ_i$.

Die Wasserstein-Approximation wiederum zieht ein coupling-Argument nach sich. Dazu Verfeinern wir das Mineka coupling (Mineka, 1973; Rösler, 1977a; 1977b) indem wir eine spezielle Block-Konstruktion einführen. Dieses Verfahren führt zu exponentiell besseren coupling-Raten für die fraglichen Summen.

Als Anwendungen werden, im Kontext quasi-logarithmischer Strukturen, ein Analogon des Fundamentallemmas und ein Analogon des Haupttheorems von Kubilius (1962, 1964) aus der probabilistischen Zahlentheorie hergeleitet. Der erste Satz gibt Konvergenzraten für die Totalvariationsdistanz der Verteilung des Spektrums der kleinen Komponenten und der Zufallsvariablen $Z_1, Z_2, \ldots$ an. Der Beweis verwendet wiederum das neue coupling-Verfahren. Der zweite Satz ist eine kombinatorische Version des zentralen Grenzwertsatzes für beschränkte Varianzen; als Spezialfall erhalten wir ein Analogon des zentralen Grenzwertsatzes von Erdős und Kac (1940).

Eine weitere Anwendung behandelt quasi-logarithmische additive arithmetische Halbgruppen. Wir zeigen, dass in diesen Halbgruppen alle Partitionsmengen asymptotische Dichte haben. Der Beweis beruht auf Resultaten von Compton (1989) und entsprechend erweiterten Resultaten von Woods (1997). Auf diese Weise erhalten wir einen logischen Grenzwertsatz für monadische Logiken zweiter Ordnung der vorhergehende Resultate von Bell und Burris (2003), Granovsky und Stark (2006) und Stark (2006) erweitert.

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Contents

1 Introduction

Quasi-logarithmic structures?

Random decomposable combinatorial structures The notion of a random decomposable combinatorial structure is, at least intuitively, best introduced by giving an example of such a structure.

Consider the set S_n of all permutations of the set $\{1, \ldots, n\}$. Each permutation $\pi \in S_n$ consists of a certain number of cycles. The cycles are permutations which cannot be decomposed further into smaller cycles, and from this point of view are "irreducible" objects into which π can be decomposed. Therefore, permutations form what is called a *decomposable combinatorial structure*. Randomness enters if we pick permutations $\pi \in S_n$ randomly. That is, we endow the (finite) set S_n with a measure, which usually is the uniform measure ν_n, so that each permutation is equally likely chosen. In this way, permutations constitute a *random* decomposable combinatorial structure.

Permutations give rise naturally to a mapping

$$C^{(n)} := \big(C_1^{(n)}, \ldots, C_n^{(n)}\big) : S_n \longrightarrow \mathbb{Z}_+^n \,,$$

where $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ and where $C_i^{(n)}(\pi)$ is defined to be the number of cycles of size i which appear in the cycle decomposition of π. The mapping $C^{(n)}$ is called, in the context of permutations, the cycle spectrum. More generally, for arbitrary decomposable combinatorial structures, it is referred to as the *component spectrum*. The component spectrum $C^{(n)}$ of a random decomposable combinatorial structure is a random vector, whose distribution with respect to the measure ν_n we denote by $\mathscr{L}_{\nu_n}(C^{(n)})$.

A main goal of the theory of random decomposable combinatorial structures is the study of the asymptotic distributional behaviour of certain functions of the component spectrum $C^{(n)}$ as $n \to \infty$. For example, one can study the limiting behaviour of the spectrum of "small" components,

$$\lim_{n\to\infty} \mathscr{L}_{\nu_n}\big(C_1^{(n)}, \ldots, C_a^{(n)}\big) = ?\,, \qquad \text{for a fixed } a \in \mathbb{N}, \tag{1.1}$$

or one can study asymptotics of the distribution of the number of components a decomposable combinatorial structure consists of,

$$\lim_{n\to\infty} \mathscr{L}_{\nu_n}\Big(\sum\nolimits_{i=1}^{n} C_i^{(n)}\Big) = ?\,. \tag{1.2}$$

From the viewpoint of probability theory the component spectrum $C^{(n)}$ has an undesirable property which causes problems when establishing limit theorems as above; its entries $C_1^{(n)}, \ldots, C_n^{(n)}$ are not independent. In fact, they satisfy a "total size conservation law"

$$\sum_{i=1}^{n} iC_i^{(n)} = n\,. \tag{1.3}$$

However, many decomposable combinatorial structures satisfy a *conditioning relation* of the form

$$\mathscr{L}_{\nu_n}\big(C^{(n)}\big) = \mathscr{L}\Big(Z_1, \ldots, Z_n \,\Big|\, \textstyle\sum_{i=1}^{n} iZ_i = n\Big)\,, \tag{1.4}$$

for at least all n large enough, where $\{Z_i\}_{i\in\mathbb{N}}$ is a sequence of *independent* random variables that take values in $\mathbb{Z}_+$. Among the structures that satisfy a conditioning relation as (1.4) are the three "classical" types of random decomposable combinatorial structures:

Assemblies: These are *labelled* random decomposable combinatorial structures, such as permutations. A permutation $\pi \in S_n$ can be chosen randomly by first partitioning the set $\{1, \ldots, n\}$ into pairwise disjoint non-empty subsets, and then by endowing each set of size i with the structure of one of the $(i-1)!$ possible cycles. Assemblies satisfy (1.4) with Poisson distributed random variables.

Multisets: These are *unlabelled* random decomposable combinatorial structures. As an example of such a structure, consider the set of all monic polynomials $f(X)$ of degree n in one indeterminate X over a finite field; any such polynomial can be written as a product of irreducible monic polynomials. A monic polynomial of degree n can be chosen randomly by first partitioning n into positive summands, and then by selecting for each summand i one of the irreducible monic polynomials of degree i. Multisets satisfy (1.4) with negative binomially distributed random variables.

Selections: These are unlabelled random decomposable combinatorial structures like multisets, with the restriction that all components of an instance of such a structure must be distinct. Square free monic polynomials in one indeterminate over a finite field belong to the class of selections. For selections (1.4) holds true with binomially distributed random variables.

In order that the conditioning relation (1.4) can be applied to study distributional limits as in (1.1) or (1.2), it is necessary first to study the distributional asymptotics of

$$\frac{1}{n}\sum_{i=1}^{n} iZ_i\,, \tag{1.5}$$

as $n \to \infty$.

In this thesis, we only consider random decomposable combinatorial structures, in which (1.5) has a special asymptotic behaviour, where $\frac{1}{n}\sum_{i=1}^{n} iZ_i$ converges in distribution to the so-called *generalized Dickman distribution*.

ESF-structures and the Dickman distribution We return to the example of permutations as random decomposable combinatorial structures. We mentioned above that the set S_n usually is endowed with the uniform probability measure ν_n. New random decomposable combinatorial structures can be constructed by "tilting" the permutations. That is, S_n is endowed with a measure, $\tilde{\nu}_n$ say, defined by

$$\tilde{\nu}_n\big(\{\pi\}\big) \propto \theta^{N(\pi)}\nu_n\big(\{\pi\}\big) \qquad \text{for all } \pi \in S_n,$$

where $\theta > 0$ is a fixed parameter and $N(\pi)$ is the number of cycles in π. In other words, a permutation π is selected randomly from S_n, with probability proportional to $\theta^{N(\pi)}$. Under this new measure, the cycle spectrum $C^{(n)}$ is distributed according to the *Ewens sampling formula* $\mathrm{ESF}_n(\theta)$,

$$\tilde{\nu}_n\big(C^{(n)} = (c_1, \ldots, c_n)\big) = \mathbf{1}\Big\{\sum\nolimits_{i=1}^{n} ic_i = n\Big\} \prod_{j=1}^{n} \frac{1+n-j}{\theta+n-j}\Big(\frac{\theta}{j}\Big)^{c_j}\frac{1}{c_j!}\,,$$

and we refer to this new random combinatorial structure as $\mathrm{ESF}(\theta)$-*structure*. The (expected) number of components of an $\mathrm{ESF}(\theta)$-structure grows as $\theta \log n$; hence, ESF-structures are an example of what is generally referred to as a *logarithmic structure*.

It can be shown that the conditioning relation (1.4) holds true with Poisson distributed random variables Z_i with expectations $\mathbb{E}Z_i = \theta/i$. The sum $\sum_{i=1}^{n} iZ_i$ then has a compound Poisson distribution $\mathrm{CP}(\{\lambda_i\}_{i\in\mathbb{N}})$ with rates $\lambda_i := \theta/i$ for $1 \le i \le n$ and $\lambda_i := 0$ for $i > n$, which we denote by $\mathrm{CP}(\theta, n)$ briefly. Its expectation and standard deviation are of order of magintude n, and the scaled sum $\frac{1}{n}\sum_{i=1}^{n} iZ_i$ can be shown to have a non-normal distributional limit as $n \to \infty$. In fact,

$$\lim_{n\to\infty} \mathscr{L}\Big(\frac{1}{n}\sum\nolimits_{i=1}^{n} iZ_i\Big) = \mathrm{GD}(\theta)\,, \tag{1.6}$$

where $\mathrm{GD}(\theta)$ is the *generalized Dickman distribution* with parameter θ. This distribution is concentrated on the positive real numbers and has a density p_θ. Moreover, it is possible to establish a local limit theorem of the form

$$\lim_{n\to\infty} n\mathbb{P}\Big[\sum\nolimits_{i=1}^{n} iZ_i = k_n\Big] = p_\theta(x)\,, \qquad \text{if } \lim_{n\to\infty} k_n/n = x > 0. \tag{1.7}$$

For $\theta = 1$ and $x > 0$ the density $p_\theta(x)$ is, up to a normalizing constant, equal to the Dickman function

$$\rho(x) = \lim_{n\to\infty} \frac{1}{n}\Big|\{1 \le i \le n : \text{the largest prime divisor of } i \text{ is } \le n^{1/x}\}\Big|,$$

well-known in probabilistic number theory.

Logarithmic structures *Logarithmic structures* can be seen as perturbations of ESF-structures; they are random decomposable combinatorial structures which satisfy the conditioning relation (1.4) with a sequence of independent random variables $\{Z_i\}_{i\in\mathbb{N}}$ such that local Dickman approximation (1.7) still remains valid for some parameter $\theta > 0$.

Arratia et al. (2003) work with a condition on the sequence $\{Z_i\}_{i\in\mathbb{N}}$ which ensures that a limit of the form (1.6) holds true, the *logarithmic condition* $\mathsf{LC}(\theta)$

$$\lim_{i\to\infty} i\mathbb{E}Z_i = \theta \qquad \text{and} \qquad \lim_{i\to\infty} i\mathbb{P}[Z_i = 1] = \theta\,; \tag{1.8}$$

this is clearly satisfied if $Z_i \sim \mathrm{Po}(\theta/i)$. In the context of the "classical" types of random decomposable structures, which satisfy (1.4) with special Poisson, negative binomially or binomially distributed random variables Z_i, the logarithmic condition (1.8) entails (1.7).

If, however, the Z_i have some arbitrary distribution, the logarithmic condition has to be strengthend in order to obtain a Dickman approximation (1.6). Arratia et al. (2003) introduce a uniformity condition, which allows the deviation of $\mathscr{L}(Z_i)$ from $\mathrm{Po}(\theta/i)$ to be accurately controlled. They write $Z_i = \sum_{j=1}^{r_i} Z_{ij}$, where $Z_{i1}, \ldots, Z_{ir_i}$ are independent and identically distributed. They then define

$$\varepsilon_{i1}(\theta, r_i) := \frac{ir_i}{\theta}\,\mathbb{P}[Z_{i1} = 1] - 1 \quad \text{and} \quad \varepsilon_{ik}(\theta, r_i) := \frac{ir_i}{\theta}\,\mathbb{P}[Z_{i1} = k] \quad \text{if } k \ge 2.$$

If the $\varepsilon_{ik}(\theta, r_i)$ are small and i is large then the distribution of Z_{i1} is close to the Bernoulli $\mathrm{Be}(\theta/(ir_i))$ distribution. If i is large, $\mathrm{Be}(\theta/(ir_i))$ in turn is close to $\mathrm{Po}(\theta/(ir_i))$, and thus $\mathscr{L}(Z_i)$ to $\mathrm{Po}(\theta/i)$. The *uniformity condition* $\mathsf{UC}^*(\theta)$, as is used by Arratia et al. (2003) or Arratia et al. (2005), then states that

$$\mu_i(\theta) := \sum_{k=1}^{\infty} k \sup_{j>i} |\varepsilon_{jk}(\theta, r_j)|$$

converges to 0 as $i \to \infty$, at some specified convergence rate. The sum $\sum_{i=0}^{n} \mu_i(\theta)$ is used to control the devation of $\mathscr{L}(\sum_{i=1}^{n} iZ_i)$ from the compound

Poisson distribution $\mathrm{CP}(\theta, n)$, defined in the previous paragraph. The questions about distributional limits posed in (1.1) and (1.2) can be answered by

$$\lim_{n\to\infty} \mathscr{L}_{\nu_n}\big(C_1^{(n)}, \ldots, C_a^{(n)}\big) = \mathscr{L}\big(Z_1, \ldots, Z_a\big)\,, \tag{1.9}$$

and by

$$\frac{\sum_{i=1}^n C_i^{(n)} - \theta \log n}{\sqrt{\theta \log n}} \xrightarrow{\mathcal{D}} \mathrm{N}(0,1)\,. \tag{1.10}$$

Note that much stronger results are achieved; these two just have expository character.

Although the framework of Arratia et al. (2003) is very powerful, in the sense that their notion of a logarithmic structure is not restricted to a specific type of random decomposable combinatorial structure such as an assembly or a multiset, it seems that the logarithmic condition (1.8) is too restrictive. Indeed, in some special situations, limiting results such as (1.10), with the same normalizing constants, can be achieved if the $i\mathbb{E}Z_i$ and $i\mathbb{P}[Z_i = 1]$ "oscillate" around the parameter θ instead of converging towards it.

Hybrid ESF-structures and Beurling-type arithmetic semigroups An early result in this direction was established by Tarakanov and Čistjakov (1975). They studied permutations without cycles of even lengths, and showed that (1.9) holds for this form of random decomposable structures with $Z_i \sim \mathrm{Po}(1/i)$ for odd i and $Z_i = 0$ for even i.

More generally, Arratia et al. (1995) introduced the notion of a *hybrid* $\mathrm{ESF}(\theta_1, \theta_2)$*-structure* as a random decomposable combinatorial structure that satisfies the conditioning relation (1.4) with Poisson distributed random variables Z_i, where $\mathbb{E}Z_i = \theta_1/i$ if i is odd, and $\mathbb{E}Z_i = \theta_2/i$ if i is even. For example, if $\theta_1 - \theta_2$ is an even integer, Arratia et al. (1995) prove a limit theorem of the form (1.9). Conditions very similar to those of Arratia et al. (1995) were also examined by Flajolet and Soria (1990). They derived results such as (1.10) for the classical types of random decomposable combinatorial structures.

Knopfmacher (1979) introduced the notion of an *additive arithmetic semigroup* as an abstract algebraic counterpart to the natural numbers $\mathbb{N}$. An additive arithmetic semigroup is a free commutative monoid A with a countable generating set, endowed with an additive $\mathbb{Z}_+$-valued norm $\|\cdot\|$ such that $A(n) = \{u \in A : \|u\| = n\}$ is finite for each $n \in \mathbb{Z}_+$. There is a close relationship between additive arithmetic semigroups and multisets. Indeed, from a combinatorial point of view every set $A(n)$ is a multiset of size n.

Under certain conditions on the counting function $a(n) := |A(n)|$, many results from analytic and probabilistic number theory carry over to the context of additive arithmetic semigroups. The classical assumption of Knopfmacher (1979) imposed on $a(n)$ turns out to be a special case of the logarithmic condition $\mathsf{LC}(\theta)$. However, inspired work of Beurling (1937) on so-called generalized integers, Zhang (1996b) greatly extended Knopfmacher's condition. Under his new assumption, the negative binomially distributed Z_i need no longer satisfy (1.8); in fact $i\mathbb{E}Z_i$ and $i\mathbb{P}[Z_i = 1]$ can exhibit oscillating behaviour around the parameter $\theta > 0$ in the form of a sum of cosine functions. Nevertheless, Zhang (1996b) proved a central limit theorem like (1.10).

It is therefore natural to ask if the logarithmic condition of Arratia et al. (2003) can be generalized in a way that allows random decomposable combinatorial structures such as hybrid ESF-structures and Beurling-type arithmetic semigroups to be incorporated as well.

The quasi-logarithmic condition In this thesis, the logarithmic condition is generalized in a way that allows us to cover these examples of random decomposable combinatorial structures. Instead of directly comparing the distribution of $\sum_{i=1}^{n} iZ_i$ with the compound Poisson distribution $\mathrm{CP}(\theta, n)$, as done by Arratia et al. (2003), the basic idea is to split this comparison into two steps.

Step 1: Setting $\theta_i := i\mathbb{E}Z_i$, the distribution of $\sum_{i=1}^{n} iZ_i$ is compared with the compound Poisson distribution $\mathrm{CP}(\{\lambda_i\}_{i\in\mathbb{N}})$ with rates $\lambda_i := \mathbb{E}Z_i = \theta_i/i$ for $1 \le i \le n$ and $\lambda_i := 0$ for $i > n$. For this, a condition similar to UC^*, as used by Arratia et al. (2003), is applied. In fact, we also write $Z_i = \sum_{j=1}^{r_i} Z_{ij}$, where $Z_{i1}, \dots, Z_{ir_i}$ are independent and identically distributed. Then we define, if $\theta_i > 0$,

$$\varepsilon_{i1}(\theta_i, r_i) := \frac{ir_i}{\theta_i}\,\mathbb{P}[Z_{i1} = 1] - 1 \quad \text{and} \quad \varepsilon_{ik}(\theta_i, r_i) := \frac{ir_i}{\theta_i}\,\mathbb{P}[Z_{i1} = k] \quad \text{if } k \ge 2.$$

If $\theta_i = 0$, we set $\varepsilon_{ik}(\theta_i, r_i) := 0$ for all $k \in \mathbb{N}$. If the $\varepsilon_{ik}(\theta_i, r_i)$ are small and i is large then the distribution of Z_{i1} is close to the Bernoulli $\mathrm{Be}(\theta_i/(ir_i))$. And for i large, $\mathrm{Be}(\theta_i/(ir_i))$ is in turn close to $\mathrm{Po}(\theta_i/(ir_i))$, and thus $\mathscr{L}(Z_i)$ to $\mathrm{Po}(\theta_i/i)$. The new uniformity condition, UC, states that

$$\mu_i := \sum_{k=1}^{\infty} k \sup_{j>i} |\varepsilon_{jk}(\theta_j, r_j)| \tag{1.11}$$

converges to 0 as $i \to \infty$. Condition UC is very mild. Even if we impose convergence rates as in Arratia et al. (2003), it is satisfied for the three classical types of combinatorial structures (assemblies, multisets and selections) as soon as $\mathbb{E}Z_i \to 0$. Moreover, if $\mathsf{LC}(\theta)$ holds, we have $\mu_i = \mathrm{O}\big(\mu_i(\theta)\big)$.

Step 2: We control the deviation of the expected values $\mathbb{E}Z_i$ from θ/i, $1 \leq i \leq n$, by an expression of the form

$$m_n n d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum\nolimits_{i=1}^{n} iZ_i\Big), \mathscr{L}\Big(\sum\nolimits_{i=1}^{n} iZ_i + 1\Big)\Big) + n\tilde{\theta}_n(m_n, \theta)\,, \tag{1.12}$$

where $m_n = \mathrm{o}(n)$ is a suitable sequence of natural numbers that also satifies $m_n \to \infty$ as $n \to \infty$, and where

$$\tilde{\theta}_n(m, \theta) := \max_{0 \leq j \leq \lfloor n/m \rfloor} \left| \frac{1}{m} \sum_{i=1}^{m} \theta_{jm+i} - \theta \right| \qquad \text{for all } m, n \in \mathbb{N}. \tag{1.13}$$

We impose a *smoothness condition* SC by requiring that the total variation distance in (1.12) converges to 0 as $n \to \infty$. Additionally, we require that $\tilde{\theta}_n(m_n, \theta)$ converges to 0 as $n \to \infty$, for every positive sequence $m_n = \mathrm{o}(n)$ that satisfies $m_n \to \infty$. We say that $\{\theta_i\}_{i\in\mathbb{N}}$ is $\mathsf{A}(\theta)$-*convergent* in this case. This convergence in an "averaging" sense is a variant of Cesàro convergence towards θ, with some large enough rate.

Combining the assumptions UC, SC and the $\mathsf{A}(\theta)$-convergence from these two steps, gives the *quasi-logarithmic condition* $\mathsf{SUQLC}(\theta)$, a generalization of the logarithmic condition. A *quasi-logarithmic structure* is defined as a random decomposable combinatorial structure that satisfies the conditioning relation (1.4) with a sequence $\{Z_i\}_{i\in\mathbb{N}}$ that satisfies condition $\mathsf{SUQLC}(\theta)$. Limit theorems such as (1.6), (1.7), (1.9) and (1.10) then carry over to this general setting.

An outline of the thesis

Section 2 In *Section 2* we introduce the quasi-logarithmic condition for sequences of independent $\mathbb{Z}_+$-valued random variables $\{Z_i\}_{i\in\mathbb{N}}$.

First, in *Subsection 2.1*, the uniformity conditions UC^* and UC, which both control the deviations of the sum $\sum_{i=1}^{n} iZ_i$ from special compound Poisson distributions, as mentioned above, are introduced and compared to each other. The new condition UC is examined in the context of the standard distributions such as the Poisson, the Binomial and the negative Binomial.

We also recall the logarithmic condition LC and examine its relation to both UC^* and UC. This leads to the uniform logarithmic condition ULC. The working conditions of Arratia et al. (2000) and Arratia et al. (2003) are recalled, both special versions of ULC.

Whereas condition UC^* is enough to yield LC, this is not the case with UC. Thus, UC can be combined with a condition that is less restrictive than the convergence assumption in the logarithmic condition LC. This leads to the notion of A-convergent real-valued sequences in *Subsection 2.2*. We show that sequences that converge in the usual sense are A-convergent, and that these in turn are convergent in the sense of Cesàro. Using results from the theory of uniformly modulo 1 distributed sequences, it is proved that periodic functions are A-convergent and that their convergence rates depend on the irrationality type of the frequency.

Replacing the notion of convergence by A-convergence in LC leads to the quasi-logarithmic condition QLC, introduced in *Section 2.3*. The combination of the conditions QLC and UC yields the uniform quasi-logarithmic condition UQLC. We show that if the random variables Z_i are either Poisson, binomially or negative binomially distributed, A-convergence of $i\mathbb{E}Z_i$ is enough to entail the uniform quasi-logarithmic condition UQLC.

Under condition UQLC, infinitely many of the $\mathbb{Z}_+$-valued random variables Z_i can be 0 almost surely, a situation which cannot happen under the logarithmic condition LC. Thus, we introduce a smoothness condition SC, which ensures that “not too many” of the Z_i are 0; for example, it prevents the sums $\sum_{i=1}^n iZ_i$ from being concentrated on subsets of $\mathbb{Z}_+$ whose elements are multiples of some integer larger than 1. The combination of SC and UQLC yields the smoothed uniform quasi-logarithmic condition SUQLC. It is shown, using coupling results from later sections, that ULC is a special case of SUQLC. At the end of the subsection, our condition SUQLC is compared with other logarithmic conditions as used by Flajolet and Soria (1990), Arratia et al. (1995), Stark (1997a) or Stark (1997b).

Section 3 In *Section 3* approximation results for the generalized Dickman distribution are established.

In *Subsection 3.1*, the compound Poisson distribution is recalled, and a brief overview of Stein's method for distributional approximation, in particular for the compound Poisson, is given. We give upper bounds for the Wasserstein distance between $\mathscr{L}(\sum_{i=1}^n iZ_i)$ and $\mathrm{CP}(\theta, n)$.

Then, in the following *Subsection 3.2*, the generalized Dickman distribution $\mathrm{GD}(\theta)$ is defined. The results from the previous subsection and Dickman approximation results of Arratia et al. (2003) are used to derive upper bounds for the Wasserstein distance between $\mathscr{L}(\frac{1}{n}\sum_{i=1}^n iZ_i)$ and $\mathrm{GD}(\theta)$. More precisely, we prove (a more general form of) the following theorem.

Theorem. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random vari-*

ables, and let $\theta > 0$. There is a constant $c > 0$ such that, for every $m, n \in \mathbb{N}$,

$$d_{\mathrm{W}}\Big(\mathscr{L}\Big(\frac{1}{n}\sum\nolimits_{i=1}^{n} iZ_i\Big), \mathrm{GD}(\theta)\Big) \le c\Big(\frac{1}{n}\sum_{i=0}^{n}\Big(\mu_i \vee \frac{1}{i}\Big)$$
$$+ m d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum\nolimits_{i=1}^{n} iZ_i\Big), \mathscr{L}\Big(\sum\nolimits_{i=1}^{n} iZ_i + 1\Big)\Big) + \frac{m}{n} + \tilde{\theta}_n(m,\theta)\Big),$$

where $\theta_i := i\mathbb{E}Z_i$, μ_i is defined in (1.11) and $\tilde{\theta}_n(m,\theta)$ in (1.13).

The Wasserstein distance converges to 0 under the smoothed uniform quasi-logarithmic condition $\mathsf{SUQLC}(\theta)$.

After proving some technical preparatory lemmas, the local approximation theorem below, or rather a more general version of it, is established in *Subsection 3.3.*

Theorem. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables that satisfies the condition $\mathsf{SUQLC}(\theta)$. Then, for any sequence $\{k_n\}_{n\in\mathbb{N}}$ of non-negative integers such that $\lim_{n\to\infty} k_n/n = x > 0$, we have*

$$\lim_{n\to\infty} n\mathbb{P}\Big[\sum\nolimits_{i=1}^{n} iZ_i = k_n\Big] = p_\theta(x)\,.$$

Section 4 *Section 4* deals with random decomposable combinatorial structures, especially quasi-logarithmic structures, and additive arithmetic semigroups.

Random decomposable combinatorial structures are introduced in *Subsection 4.1*, as stochastic processes $\{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$, where Ω_n is a finite set endowed with some probability measure ν_n, and where $C^{(n)}$ is a $\mathbb{Z}_+^n$-valued random vector that satisfies a total size conservation law as in (1.3). We also introduce tilted structures. Our notion of tilting extends the one used in Arratia et al. (2003); it allows one to bias the structure not just with respect to the number of components, but with respect to the number of components *of each size* that appear in the decomposition of the structure. We formally introduce the conditioning relation (1.4) and examine its behaviour under tilting. At the end of the subsection we recall the definitions of classical random decomposable combinatorial structures such as assemblies, multisets and selections. Finally, hybrid ESF-structures, in a form that generalizes the original definition of Arratia et al. (1995), are introduced as tilted random permutations.

In *Subsection 4.2*, quasi-logarithmic structures are introduced as random decomposable structures for which the conditioning relation holds with a sequence $\{Z_i\}_{i\in\mathbb{N}}$ for which the smoothed uniform quasi-logarithmic condition SUQLC

holds true. Properties of quasi-logarithmic assemblies, multisets and selections are examined.

Related to multisets, additive arithmetic semigroups are the topic of *Subsection 4.3*. Quasi-logarithmic additive arithmetic semigroups are defined via quasi-logarithmic multisets. We investigate the relation of the number of prime elements (components) of norm n, $p(n)$, to the number of all elements of norm n, $a(n)$, of an arithmetic semigroup, in the quasi-logarithmic context. In fact, we show that "Beurling-type" additive arithmetic semigroups, as introduced by Zhang (1996b) by giving conditions on $a(n)$, are quasi-logarithmic, by using abstract prime element theorems of Zhang (1996a). We prove the following abstract "inverse" prime element theorem.

Theorem. *Let A be an additive arithmetic semigroup with counting function $a(n)$ and prime element counting function $p(n)$. Let $\theta > 0$ and $q > 1$. Then it follows that*

$$p(n) \sim \theta q^n n \quad \Rightarrow \quad a(n) \sim c q^n n^{\theta-1} \ell(n) \,,$$

where $c > 0$ is a constant and $\ell(n)$ a slowly varying function (and both can be given explicitly).

Although this result is used in this thesis only to establish logical limit laws in a later section, it has some interest on its own, since it refines an "inverse" prime element theorem of Knopfmacher and Warlimont (2002), and generalizes similar results of Arratia et al. (2003) and Arratia et al. (2005).

Section 5 The applications that are given in *Section 5* can be divided into two parts. The first deals with so-called asymptotic density and partition sets in additive arithmetic semigroups and its relation to logical limit laws in model theory, addressing a problem posed in Burris (2001). The second part deals with distributional limit theorems like (1.1) and (1.2).

In *Subsection 5.1* we give a brief introduction into logical limit laws, and we explain how additive arithmetic semigroups are associated with classes of finite L-structures, where L is some finite purely relational language. We also introduce the notion of asymptotic density and of partition sets. The main result of the subsection is the following theorem.

Theorem. *In a quasi-logarithmic additive arithmetic semigroup all partition sets have asymptotic density.*

The proof relies on the abstract inverse prime element theorem mentioned above, the density theorem of Compton (1989) and an extension of the density theorem of Woods (1997). In order to establish the latter result, we need to extend in turn a Tauberian theorem of Woods (1997). An immediate consequence, invoking another result of Compton (1989), is then the following logical limit result.

Theorem. *Let* L *be a finite, purely relational language, and let* $\mathcal{K}$ *be an adequate class of finite* L*-structures, such that the associated additive arithmetic semigroup* $\mathcal{A}_{\mathcal{K}}$ *is quasi-logarithmic. Then* $\mathcal{K}$ *has a monadic second-order limit law.*

We then compare these theorems with related results of Bell and Burris (2003), Granovsky and Stark (2006) and Stark (2006). The results of this subsection can also be found in Nietlispach (2007b).

In *Subsection 5.2* we return to general quasi-logarithmic structures. We first prove simple distributional limit theorems for the length of the largest component and for the spectrum of small components. The result on the spectrum of small components is improved, by incorporating total variation distance as follows.

Theorem. *Let* $\{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ *be a quasi-logarithmic structure that satisfies the conditioning relation with a sequence* $\{Z_i\}_{i\in\mathbb{N}}$ *of independent* $\mathbb{Z}_+$*-valued random variables. It follows that*

$$\lim_{n\to\infty} d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C_1^{(n)}, \dots, C_{a_n}^{(n)}), \mathscr{L}(Z_1, \dots, Z_{a_n})\big) = 0$$

for every positive integer sequence $a_n = \mathrm{o}(n)$*. Moreover, under the additional assumptions on the sequence* $\{Z_i\}_{i\in\mathbb{N}}$ *that there are constants* $\alpha_1, \alpha_2, \alpha_3 > 0$ *such that*

$$\mu_i = \mathrm{O}\Big(\frac{1}{i^{\alpha_1}}\Big), \tag{1.14a}$$

$$\tilde{\theta}_n(m_n, \theta) = \mathrm{O}\Big(\frac{1}{m_n^{\alpha_2}}\Big), \tag{1.14b}$$

$$d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum\nolimits_{i=a_n+1}^{n} iZ_i\Big), \mathscr{L}\Big(\sum\nolimits_{i=a_n+1}^{n} iZ_i + 1\Big)\Big) = \mathrm{O}\Big(\Big(\frac{a_n}{n}\Big)^{\alpha_3}\Big), \tag{1.14c}$$

recalling that $\theta_i := i\mathbb{E}Z_i$ *and that* μ_i *and* $\tilde{\theta}_n(m_n, \theta)$ *are defined in* (1.11) *and* (1.13)*, respectively, it follows that*

$$d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C_1^{(n)}, \dots, C_{a_n}^{(n)}), \mathscr{L}(Z_1, \dots, Z_{a_n})\big) = \mathrm{O}\Big(\Big(\frac{a_n}{n}\Big)^{\alpha}\Big), \quad \textit{for some } \alpha > 0.$$

The first two assumptions, (1.14a) and (1.14b) are very mild. The first assumption is satisfied by all assemblies, multisets and selections, the second holds true, for example, if $\theta_i = i\mathbb{E}Z_i$ oscillates around θ in the form of a sinusoidal function or converges towards θ fast enough (hybrid ESF-structures and Beurling type additive arithmetic semigroups exhibit such a behaviour of θ_i). The third restriction, (1.14c), is more interesting; it is examined in Section 6. Note that the theorem above is an analogue of the *Kubilius fundamental lemma* in number theory (Kubilius, 1964).

The second part of Subsection 5.2 deals with additive functions on random decomposable combinatorial structures. We prove an analogue of the *Kubilius main theorem* (Kubilius, 1962) which generalizes previous results of Flajolet and Soria (1990), Zhang (1996b, 2002), and Arratia et al. (2003, 2005). Straightforward corollaries are analogues of the *Kubilis-Shapiro central limit theorem* and the *Erdős-Kac theorem* (cf. Elliott (1980, Chapter 12)). The following special case of the Erdős-Kac theorem on the number of components in a quasi-logarithmic structure gives an idea of what is obtained.

Theorem. *Let $\{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ be quasi-logarithmic. Then there is a $\theta > 0$ such that*

$$\frac{\sum_{i=1}^n C_i^{(n)} - \theta \log n - \log \ell(n)}{\sqrt{\theta \log n}} \xrightarrow{\mathcal{D}} \mathrm{N}(0,1)\,,$$

where $\ell(n)$ is slowly varying at infinity.

The results of this subsection can also be found in Nietlispach (2007a) and Nietlispach (2007c).

Section 6 In *Section 6* we examine under what conditions a sequence $\{Z_i\}_{i\in\mathbb{N}}$ that satisfies the uniform quasi-logarithmic condition UQLC also satisfies the smoothness condition SC in the form

$$d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum\nolimits_{i=a_n+1}^n iZ_i\Big), \mathscr{L}\Big(\sum\nolimits_{i=a_n+1}^n iZ_i + 1\Big)\Big) = \mathrm{O}\Big(\Big(\frac{a_n}{n}\Big)^{\alpha}\Big)\,,$$

where $0 < \alpha < 1$, a version that is used for the convergence rates in the Dickman approximation and in the analogue of the Kubilius fundamental lemma above.

The usual approach to bound the total variation distance between the distributions $\mathscr{L}(\sum_{i=a_n+1}^n iZ_i)$ and $\mathscr{L}(\sum_{i=a_n+1}^n iZ_i + 1)$, where we have sums of independent but not identically distributed random variables, is to apply Mineka coupling (Mineka, 1973; Rösler, 1977a; 1977b). However, in our context where $\{Z_i\}_{i\in\mathbb{N}}$ satisfies the condition UQLC, Mineka coupling only yields poor bounds of the form $(\log(n/a_n))^{-\beta}$, for some $\beta > 0$. We outline this problem in *Subsection 6.1*, and we sketch an alternative coupling method which in our situation yields exponentially sharper coupling rates.

In *Subsections 6.2*, *6.3* and *6.4*, we establish the coupling in two situations, distinguishing two kinds of subsets $I \subset \mathbb{N}$ where

$$\mathbb{P}[Z_i = 0] \geq \psi_0 \qquad \text{and} \qquad \mathbb{P}[Z_i = 1] \geq \frac{\psi_1}{i} \qquad \text{for all } i \in I,$$

for some $\psi_0, \psi_1 > 0$. Combining result from both subsections yields the following theorem.

Theorem. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables which satisfies the uniformity condition* UC*. Assume that $\theta_i = i\mathbb{E}Z_i$, $i \in \mathbb{N}$, is the integer skeleton of the sinusoidal function*

$$\theta + \sum_{l=1}^{N} \lambda_l \cos(2\pi f_l x - \varphi_l)\,, \qquad x \in \mathbb{R},$$

with $N \in \mathbb{N}$, $\theta > 0$, amplitudes $\lambda_l > 0$ such that $\sum_{l=1}^{N} \lambda_l \leq \theta$, frequencies $f_l > 0$ and phases $0 \leq \varphi_l < 2\pi$. Let $a_n = \mathrm{o}(n)$ be a non-negative integer sequence. If each Z_i is concentrated on the even integers, we have

$$d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum\nolimits_{i=a_n+1}^{n} iZ_i\Big), \mathscr{L}\Big(\sum\nolimits_{i=a_n+1}^{n} iZ_i + 1\Big)\Big) = 1 \qquad \textit{for all } n \in \mathbb{N}.$$

If not, there is a $0 < \alpha < 1$ such that

$$d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum\nolimits_{i=a_n+1}^{n} iZ_i\Big), \mathscr{L}\Big(\sum\nolimits_{i=a_n+1}^{n} iZ_i + 1\Big)\Big) = \mathrm{O}\Big(\Big(\frac{a_n}{n}\Big)^{\alpha}\Big)\,.$$

The coupling of Subsection 6.3 can also be found in Nietlispach (2007a), where also much more general situations are considered. The coupling in Subsection 6.4 is developed in Nietlispach (2007c).

Notation

The following notation is used throughout the thesis. We use $\mathbb{N} := \{1, 2, 3, \dots\}$ for the natural numbers, $\mathbb{Z}$ for the integers, $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ for the non-negative integers, $\mathbb{R}$ for the real numbers and $\mathbb{R}_+ := [0, \infty)$ for the non-negative real numbers.

For $x, y \in \mathbb{R}$ we use $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$. We also set $x^+ := x \vee 0$ and $\bar{x} := x \vee 1$. Moreover, $\lfloor x \rfloor$ is the largest integer smaller than or equal to x, $\lceil x \rceil$ is the smallest integer larger than or equal to x, and $\langle x \rangle$ is the distance of x to its nearest integer. If $x > 0$, we denote the natural logarithm by $\log x$.

Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two real-valued sequences. We use $x_n \sim y_n$ to describe the asymptotic equivalence $\lim_{n\to\infty} x_n/y_n = 1$, and $x_n = \mathrm{o}(y_n)$ for $\lim_{n\to\infty} x_n/y_n = 0$. If $\limsup_{n\in\mathbb{N}}|x_n/y_n| < \infty$ we use either $x_n = \mathrm{O}(y_n)$ or, equivalently, the Vinogradov notation $x_n \ll y_n$. If $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ have the same order of magnitude, that is if $x_n = \mathrm{O}(y_n)$ and $y_n = \mathrm{O}(x_n)$, we write $x_n \asymp y_n$.

Sums over empty domains are set to be equal to 0, whereas products over empty domains are defined to be 1.

Let A be a set. Then $\mathbf{1}\{a \in A\}$ is defined to be 1 if $a \in A$, and 0 else.

Let X and $\{X_n\}_{n\in\mathbb{N}}$ be random variables. Then $\mathscr{L}(X)$ denotes the distribution of X. If $\{X_n\}_{n\in\mathbb{N}}$ converges to X in distribution, we write $X_n \xrightarrow{\mathcal{D}} X$; $\{X_n\}_{n\in\mathbb{N}}$ converges to X in probability, we use $X_n \xrightarrow{\mathcal{P}} X$. Assume that X has some specific distribution, i. e. the standard normal $\mathrm{N}(0,1)$. Then we express this by $X \sim \mathrm{N}(0,1)$.

Further notation can be found in Appendix A.1.

2 The quasi-logarithmic condition

2.1 Two uniformity conditions

We first introduce and discuss two uniformity conditions, UC^* and UC, on sums of independent $\mathbb{Z}_+$-valued random variables, which are later used to control the deviation of the distribution of these sums from special compound Poisson distributions. The construction of UC^* is due to Arratia et al. (2003, Section 7.2)). The second condition, UC, is a modification of UC^* first used in Nietlispach (2007a). We examine the relationship of UC^* and UC to the logarithmic condition LC.

2.1.1 Conditions UC^* and UC

Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of $\mathbb{Z}_+$-valued independent random variables. Let $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ be a sequence of natural numbers, and let

$$\mathcal{Z}(\mathfrak{r}) := \{Z_{ij} : i \in \mathbb{N} \text{ and } 1 \le j \le r_i\} \tag{2.1}$$

be a family of independent $\mathbb{Z}_+$-valued random variables, such that $Z_{i1}, \ldots, Z_{ir_i}$ are identically distributed for each $i \in \mathbb{N}$, and such that

$$Z_i = \sum_{j=1}^{r_i} Z_{ij} \qquad \text{for all } i \in \mathbb{N}. \tag{2.2}$$

For $x \in \mathbb{R}_+ \cup \{\infty\}$ and $r \in \mathbb{N}$, we define

$$\varepsilon_{ik}(x, r) := \begin{cases} \dfrac{ir}{x}\,\mathbb{P}[Z_{i1} = k] - \mathbf{1}\{k = 1\} & \text{if } x \in (0, \infty] \text{ and } k \in \mathbb{N}, \\ 0 & \text{if } x = 0 \text{ and } k \in \mathbb{N}. \end{cases} \tag{2.3}$$

For any sequence $\Theta := \{\theta_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_+ \cup \{\infty\}$ with support

$$I := \{i \in \mathbb{N} : \theta_i > 0\}, \tag{2.4}$$

we then set

$$\mu_i(\Theta, \mathfrak{r}) := \sum_{k=1}^{\infty} k \sup_{j>i} |\varepsilon_{jk}(\theta_j, r_j)| \qquad \text{for all } i \in \mathbb{Z}_+.$$

Clearly, $\{\mu_i(\Theta,\mathfrak{r})\}_{i\in\mathbb{N}}$ is a non-negative, monotonically decreasing sequence. If $\mu_i(\Theta,\mathfrak{r})$ is close to 0 then the distribution of Z_{i1} is close to the Bernoulli $\mathrm{Be}(\theta_i/(ir_i))$-distribution, at least for large i. But then in turn $\mathrm{Be}(\theta_i/(ir_i))$ is close to $\mathrm{Po}(\theta_i/(ir_i))$, and thus the distribution of Z_i to $\mathrm{Po}(\theta_i/i)$.

We are interested in two special versions of the sequence $\Theta = \{\theta_i\}_{i\in\mathbb{N}}$, and we use simplified notations for $\mu_i(\Theta,\mathfrak{r})$ in these cases, namely

$$\mu_i(\theta,\mathfrak{r}) := \mu_i(\Theta,\mathfrak{r}) \quad \text{if } \theta_i = \theta \in (0,\infty) \text{ for all } i \in \mathbb{Z}_+, \tag{2.5}$$

$$\mu_i(\mathfrak{r}) := \mu_i(\Theta,\mathfrak{r}) \quad \text{if } \theta_i = i\mathbb{E}Z_i \text{ for all } i \in \mathbb{Z}_+. \tag{2.6}$$

Definition 2.1. The sequence $\{Z_i\}_{i\in\mathbb{N}}$ satisfies the *uniformity condition* $\mathsf{UC}^*(\theta,\mathfrak{r})$ for a constant $\theta > 0$ and a positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ if

$$\lim_{i\to\infty} \mu_i(\theta,\mathfrak{r}) = 0\,. \tag{2.7}$$

Definition 2.2. The sequence $\{Z_i\}_{i\in\mathbb{N}}$ satisfies the *uniformity condition* $\mathsf{UC}(\mathfrak{r})$ for a positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ if

$$\lim_{i\to\infty} \mu_i(\mathfrak{r}) = 0\,. \tag{2.8}$$

Hence, condition UC^* controls the deviation of $\mathscr{L}(Z_i)$ from the Poisson $\mathrm{Po}(\theta/i)$-distribution, whereas UC only controls the deviation of $\mathscr{L}(Z_i)$ from a Poisson distribution which has the same expectation as Z_i.

Lemma 2.3. *For any constant $\theta > 0$ and any positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ the following equivalence is valid:*

$$\{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{UC}^*(\theta,\mathfrak{r}) \qquad \Leftrightarrow \qquad \begin{array}{l} \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{UC}(\mathfrak{r}), \\ \textit{and } \lim_{i\to\infty} i\mathbb{E}Z_i = \theta. \end{array}$$

Moreover, if $\lim_{i\to\infty} i\mathbb{E}Z_i = \theta$, *then* $\mu_i(\mathfrak{r}) = \mathrm{O}\big(\mu_i(\theta,\mathfrak{r})\big)$.

Proof. First, we assume $\mathsf{UC}^*(\theta,\mathfrak{r})$. For every $i \in \mathbb{N}$ we have

$$\begin{aligned} \mu_i(\theta,\mathfrak{r}) &= \sum_{k=1}^{\infty} k \sup_{j>i} |\varepsilon_{jk}(\theta,r_j)| \ge \sup_{j>i} \left| \sum_{k=1}^{\infty} k\varepsilon_{j,k}(\theta,r_j) \right| \\ &= \sup_{j>i} \left| \frac{jr_j}{\theta} \sum_{k=1}^{\infty} k\mathbb{P}[Z_{j1} = k] - 1 \right| = \sup_{j>i} \left| \frac{j\mathbb{E}Z_j}{\theta} - 1 \right|, \end{aligned}$$

which yields $i\mathbb{E}Z_i \to \theta$. But then there is an $i_0 \in \mathbb{N}$ such that $\theta/2 \le i\mathbb{E}Z_i \le 2\theta$ for all $i \ge i_0$. For any such i it follows that

$$\begin{aligned}
\mu_i(\mathfrak{r}) &= \sup_{j>i}\Big|\frac{jr_j}{j\mathbb{E}Z_j}\mathbb{P}[Z_{j1}=1]-1\Big| + \sum_{k=2}^{\infty} k \sup_{j>i}\frac{jr_j}{j\mathbb{E}Z_j}\mathbb{P}[Z_{j1}=k] \\
&\le \sup_{j>i}\frac{\theta}{j\mathbb{E}Z_j}\sup_{j>i}\Big|\frac{jr_j}{\theta}\mathbb{P}[Z_{j1}=1]-1\Big| + \sup_{j>i}\Big|\frac{\theta}{j\mathbb{E}Z_j}-1\Big| \\
&\quad + \sup_{j>i}\frac{\theta}{j\mathbb{E}Z_j}\sum_{k=2}^{\infty} k\sup_{j>i}\frac{jr_j}{\theta}\mathbb{P}[Z_{j1}=k] \\
&\le \sup_{j>i}\frac{\theta}{j\mathbb{E}Z_j}\,\mu_i(\theta,\mathfrak{r}) + \sup_{j>i}\frac{\theta}{j\mathbb{E}Z_j}\sup_{j>i}\Big|1-\frac{j\mathbb{E}Z_j}{\theta}\Big| \\
&\le 2\sup_{j>i}\frac{\theta}{j\mathbb{E}Z_j}\,\mu_i(\theta,\mathfrak{r}) \\
&\le 4\,\mu_i(\theta,\mathfrak{r})\,.
\end{aligned}$$

Thus, $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{UC}(\mathfrak{r})$.

Now assume that condition $\mathsf{UC}(\mathfrak{r})$ holds and that $i\mathbb{E}Z_i \to \theta$. Similarly as above, we obtain for all $i \ge i_0$ that

$$\mu_i(\theta,\mathfrak{r}) \le 2\,\mu_i(\mathfrak{r}) + \sup_{j>i}\Big|\frac{j\mathbb{E}Z_j}{\theta}-1\Big|\,,$$

which entails condition $\mathsf{UC}^*(\theta,\mathfrak{r})$.

The second part of the lemma is immediate from the calculation above. □

Corollary 2.4. *For any constant $\theta > 0$ and any positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$*

$$\sum_{i=1}^{\infty}\frac{\mu_i(\theta,\mathfrak{r})}{i} < \infty \qquad \Rightarrow \qquad \sum_{i=1}^{\infty}\frac{\mu_i(\mathfrak{r})}{i} < \infty\,. \tag{2.9}$$

The converse implication does not hold.

Proof. The left hand side of (2.9) implies that $\mu_i(\theta,\mathfrak{r}) \to 0$, because the $\mu_i(\theta,\mathfrak{r})$ are non-negative. Lemma 2.3 then yields $i\mathbb{E}Z_i \to \theta$ and $\mu_i(\mathfrak{r}) = \mathrm{O}\big(\mu_i(\theta,\mathfrak{r})\big)$, which entails (2.9). Example 2.39 below shows that the converse implication is not true. □

2.1.1.1 Condition UC^* in the sense of Arratia, Barbour and Tavaré

We consider two variants of the uniformity condition $\mathsf{UC}^*(\theta,\mathfrak{r})$ as used by Arratia et al. (2000, 2003, 2005). Under their working assumptions, $\mathsf{UC}^*(\theta,\mathfrak{r})$ is satisfied in the form

$$\sum_{i=1}^{\infty} \frac{\mu_i(\theta, \mathfrak{r})}{i} < \infty\,. \tag{2.10}$$

In Arratia et al. (2000, 2005) it is assumed that, for some $\theta > 0$, there are a positive, monotonically decreasing sequence $\{e_i(\theta)\}_{i\in\mathbb{N}}$ and a positive sequence $\{c_k(\theta)\}_{k\in\mathbb{N}}$ that satisfy

$$\lim_{i\to\infty} e_i(\theta) = 0 \qquad \text{and} \qquad \sum_{i=1}^{\infty} \frac{e_i(\theta)}{i} < \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} k c_k(\theta) < \infty\,,$$

such that

$$|\varepsilon_{ik}(\theta, 1)| \le e_i(\theta) c_k(\theta) \qquad \text{for all } i, k \in \mathbb{N},$$

where $\varepsilon_{ik}(\theta, 1)$ is defined as in (2.3) with $r_i = 1$ for all $i \in \mathbb{N}$. Under these assumptions, (2.10) and thus condition $\mathsf{UC}^*(\theta, \mathfrak{r})$, holds true, with $\mathfrak{r} = \{r_i\}_{i\in\mathbb{N}}$ and $r_i = 1$ for all $i \in \mathbb{N}$.

Arratia et al. (2003, Section 7.3) give strengthened verion of $\mathsf{UC}^*(\theta, \mathfrak{r})$ in form of three assumptions. Indeed, they assume that, for some $\theta > 0$ and some sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ of natural numbers,

Condition $(\mathbf{A}_m)$: $\varepsilon_{i1}(\theta, r_i) = \mathrm{O}(i^{-g_1})$ for some $g_1 > m$,

Condition $(\mathbf{D}_m)$: $|\varepsilon_{i1}(\theta, r_i) - \varepsilon_{i+1,1}(\theta, r_i)| = \mathrm{O}(i^{-g_2})$ for some $g_2 > m$,

Condition $(\mathbf{B}_{mn})$: for $k \ge 2$, $k\varepsilon_{ik}(\theta, r_i) \le C i^{-a_1} k^{-a_2}$ for some fixed $C > 0$, $a_1 > m$ and $a_2 > n$.

Note that $(\mathbf{A}_0)$ and $(\mathbf{B}_{01})$ are the weakest conditions of this kind to imply (2.10) and thus, via (2.7), condition $\mathsf{UC}^*(\theta, \mathfrak{r})$. Typical working conditions in Arratia et al. (2003) are $(\mathbf{A}_0)$, $(\mathbf{D}_1)$ and $(\mathbf{B}_{01})$, or $(\mathbf{A}_0)$, $(\mathbf{D}_1)$ and $(\mathbf{B}_{11})$. Other results are established under the condition $\mathsf{UC}^*(\theta, \mathfrak{r})$ directly, without further assumptions (cf. Arratia et al. (2003, Section 12.1)).

2.1.2 Condition UC for standard distributions on $\mathbb{Z}_+$

If the random variables $\{Z_i\}_{i\in\mathbb{N}}$ are all either Poisson, binomially or negative binomially distributed, condition UC is valid under very mild assumptions.

Lemma 2.5. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent Poisson distributed random variables. Then we have for any positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$*

$$\mu_i(\mathfrak{r}) \le \sup_{j>i} \frac{\mathbb{E}Z_j}{r_j} + \exp\Bigl(\sup_{j>i} \frac{\mathbb{E}Z_j}{r_j}\Bigr) - 1 \qquad \textit{for all } i \in \mathbb{N}.$$

In particular, condition $\mathsf{UC}(\mathfrak{r})$ is satisfied if $\lim_{i\to\infty} \mathbb{E}Z_i = 0$.

Proof. Because the Poisson distribution is infinitely divisible, we can choose an arbitrary positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ such that (2.2) holds, and consider a family of random variables $\mathcal{Z}(\mathfrak{r})$ as in (2.1). Then $Z_{i1} \in \mathcal{Z}(\mathfrak{r})$ is Poisson distributed with expectation $\mathbb{E}Z_i/r_i$ for each $i \in \mathbb{N}$. We recall from (2.4), using $\theta_i := i\mathbb{E}Z_i$ for all $i \in \mathbb{N}$, that $I = \{i \in \mathbb{N} : \mathbb{E}Z_i > 0\}$. Then we have

$$|\varepsilon_{j1}(j\mathbb{E}Z_j, r_j)| = 1 - \mathrm{e}^{-\mathbb{E}Z_j/r_j} \le \frac{\mathbb{E}Z_j}{r_j} \qquad \text{for all } j \in I,$$

and, for $k \ge 2$,

$$|\varepsilon_{jk}(j\mathbb{E}Z_j, r_j)| = \mathrm{e}^{-\mathbb{E}Z_j/r_j}\frac{(\mathbb{E}Z_j/r_j)^{k-1}}{k!} \le \frac{1}{k}\frac{(\mathbb{E}Z_j/r_j)^{k-1}}{(k-1)!} \qquad \text{for all } j \in I.$$

By definition, $\varepsilon_{jk}(j\mathbb{E}Z_j, r_j) = 0$ for all $k \in \mathbb{N}$, if $j \in \mathbb{N} \setminus I$. Hence, we have for every $i \in \mathbb{N}$

$$\mu_i(\mathfrak{r}) \le \sup_{j>i} \mathbb{E}Z_j/r_j + \sum_{k=2}^{\infty} \frac{(\sup_{j>i} \mathbb{E}Z_j/r_j)^{k-1}}{(k-1)!},$$

which proves the lemma. □

Lemma 2.6. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent binomially distributed random variables, $Z_i \sim \mathrm{Bin}(r_i, p_i)$, where $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ is a sequence of natural numbers and $0 \le p_i \le 1$ for all $i \in \mathbb{N}$. Then*

$$\mu_i(\mathfrak{r}) = 0 \qquad \textit{for all } i \in \mathbb{N},$$

and condition $\mathsf{UC}(\mathfrak{r})$ is satisfied without further restrictions.

Proof. The random variable $Z_{i1} \in \mathcal{Z}(\mathfrak{r})$ is Bernoulli $\mathrm{Be}(p_i)$-distributed for each $i \in \mathbb{N}$. We have from (2.4), setting $\theta_i := i\mathbb{E}Z_i = ir_ip_i$ for all $i \in \mathbb{N}$, that $I = \{i \in \mathbb{N} : p_i > 0\}$. It follows that

$$\varepsilon_{i1}(i\mathbb{E}Z_i, r_i) = \frac{ir_i}{i\mathbb{E}Z_i}\mathbb{P}[Z_{i1} = 1] - 1 = 0 \qquad \text{for all } i \in I,$$

and, for $k \ge 2$, because $\mathbb{P}[Z_{i1} = k] = 0$ in this case,

$$\varepsilon_{ik}(i\mathbb{E}Z_i, r_i) = \frac{ir_i}{i\mathbb{E}Z_i}\mathbb{P}[Z_{i1} = k] = 0 \qquad \text{for all } i \in I.$$

If $i \in \mathbb{N} \setminus I$, then $\varepsilon_{ik}(i\mathbb{E}Z_i, r_i) = 0$ for all $k \in \mathbb{N}$ by definition. Therefore, $\mu_i(\mathfrak{r}) = 0$ for all $i \in \mathbb{N}$, and the lemma follows. □

Lemma 2.7. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent negative binomially distributed random variables, $Z_i \sim \mathrm{NB}(r_i, p_i)$, where $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ is a sequence of natural numbers and $0 \le p_i < 1$ for all $i \in \mathbb{N}$. We have*

$$\mu_i(\mathfrak{r}) \le \frac{4\sup_{j>i} p_j}{1-2\sup_{j>i} p_j} \qquad \textit{for all } i \in \mathbb{N} \textit{ with } \sup_{j>i} p_j < 1/2.$$

Condition $\mathsf{UC}(\mathfrak{r})$ *is satisfied if* $\lim_{i\to\infty} p_i = 0$*, und thus if* $\lim_{i\to\infty} \mathbb{E}Z_i = 0$*.*

Proof. The random variable $Z_{i1} \in \mathcal{Z}(\mathfrak{r})$ is geometrically $\mathrm{Ge}(p_i)$-distributed for each $i \in \mathbb{N}$. Setting $\theta_i := i\mathbb{E}Z_i = ir_ip_i/(1-p_i)$ for all $i \in \mathbb{N}$, it follows from (2.4) that $I = \{i \in \mathbb{N} : p_i > 0\}$. We have

$$\varepsilon_{j1}(j\mathbb{E}Z_j, r_j) = \frac{jr_j}{j\mathbb{E}Z_j}\mathbb{P}[Z_{j1} = 1] - 1 = p_j(p_j - 2) \qquad \text{for all } j \in I,$$

and thus

$$|\varepsilon_{j1}(j\mathbb{E}Z_j, r_j)| \le 2p_j \qquad \text{for all } j \in I.$$

If $k \ge 2$, then

$$\varepsilon_{jk}(j\mathbb{E}Z_j, r_j) = \frac{jr_j}{j\mathbb{E}Z_j}\mathbb{P}[Z_{j1} = k] = p_j^{k-1}(1-p_j)^2 \le p_j^{k-1} \qquad \text{for all } j \in I.$$

Once more we note that $\varepsilon_{jk}(j\mathbb{E}Z_j, r_j) = 0$ for all $k \in \mathbb{N}$ if $j \in \mathbb{N} \setminus I$. It follows for any $i \in \mathbb{N}$ with $\sup_{j>i} p_j < 1/2$ that

$$\begin{aligned}\mu_i(\mathfrak{r}) &\le 2\sup_{j>i} p_j + \sum_{k=1}^{\infty}\big((k+1)^{1/k}\sup_{j>i} p_j\big)^k \\ &\le 2\sup_{j>i} p_j + \sum_{k=1}^{\infty}\big(2\sup_{j>i} p_j\big)^k \\ &= 2\sup_{j>i} p_j + \frac{2\sup_{j>i} p_j}{1-2\sup_{j>i} p_j}\,.\end{aligned}$$

This proves the lemma. □

2.1.3 Condition ULC

In this subsection we briefly recall the logarithmic condition, LC, as found in Arratia et al. (2003, p. 65), and consider its relation with the uniformity conditions UC* and UC.

Definition 2.8. A sequence $\{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables satisfies the *logarithmic condition* $\mathsf{LC}(\theta)$ for a constant $\theta > 0$ if

$$\lim_{i\to\infty} i\mathbb{E}Z_i = \theta\,, \tag{2.11}$$

and if

$$\lim_{i\to\infty} i\mathbb{P}[Z_i = 1] = \theta\,. \tag{2.12}$$

Clearly, if (2.11) holds, then (2.12) is equivalent to

$$\lim_{i\to\infty} i(\mathbb{E}Z_i - \mathbb{P}[Z_i = 1]) = 0\,. \tag{2.13}$$

Definition 2.9. A sequence $\{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables satisfies the *uniform logarithmic condition* $\mathsf{ULC}(\theta,\mathfrak{r})$ for a constant $\theta > 0$ and a positive integer sequence $\mathfrak{r}$ if it satisfies $\mathsf{LC}(\theta)$ and the uniformity condition $\mathsf{UC}(\mathfrak{r})$.

Lemma 2.10. *For any $\theta > 0$ and any positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ we have*

$$\begin{array}{lcl} \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{UC}^*(\theta,\mathfrak{r}) & \Leftrightarrow & \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{ULC}(\theta,\mathfrak{r}) \\ & \Leftrightarrow & \begin{array}{l} \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{UC}(\mathfrak{r}), \\ \textit{and } \lim_{i\to\infty} i\mathbb{E}Z_i = \theta. \end{array} \\ & \Rightarrow & \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{LC}(\theta). \end{array}$$

Proof. We note that $\mathsf{UC}(\mathfrak{r})$ entails $\varepsilon_{i1}(i\mathbb{E}Z_i, r_i) \to 0$. But $i\mathbb{E}Z_i \to \theta$ and (A.9) yield (2.13), which then is equivalent to (2.12). The lemma now follows from this remark and Lemma 2.3. □

We see that the logarithmic condition $\mathsf{LC}(\theta)$ is encapsulated in the uniformity condition $\mathsf{UC}^*(\theta,\mathfrak{r})$ already. This is not the case with condition $\mathsf{UC}(\mathfrak{r})$. Thus, this condition can be combined with a version of the logarithmic condition which is less strict. In fact, the quasi-logarithmic condition, which is defined later, arises from the logarithmic condition $\mathsf{LC}(\theta)$ by weakening the convergence assumption (2.11),

$$\lim_{i\to\infty} i\mathbb{E}Z_i = \theta\,.$$

Instead of requiring that $i\mathbb{E}Z_i$ converges to θ in the usual sense, we only presuppose that this sequence converges towards θ in some "averaging sense",

$$\operatorname*{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta\,.$$

2.2 A notion of convergence in average

In this section, we introduce a special form of convergence, denoted as A-convergence, for arbitrary real-valued sequences. It lies between the usual form of convergence and convergence in the sense of Cesàro:

$$\text{convergence} \quad \Rightarrow \quad \mathsf{A}\text{-convergence} \quad \Rightarrow \quad \text{Cesàro convergence.}$$

2.2.1 A_0-convergent and A-convergent sequences

For a real-valued sequence $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ and an $x \in \mathbb{R}$ we define

$$\tilde{x}_n(m,x) := \max_{0\le j\le \lfloor n/m\rfloor} \left| \frac{1}{m}\sum_{i=1}^{m} x_{jm+i} - x \right| \qquad \text{for all } m,n\in\mathbb{N}. \tag{2.14}$$

Definition 2.11. Let $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ be a real-valued sequence and let $x \in \mathbb{R}$.

(i) Let $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ be a sequence of natural numbers. The sequence $\mathfrak{x}$ is $\mathsf{A}_0(\mathfrak{m},x)$-*convergent* if $\lim_{n\to\infty} \tilde{x}_n(m_n,x) = 0$.

(ii) The sequence $\mathfrak{x}$ is $\mathsf{A}_0(x)$-*convergent* if there is a sequence $\mathfrak{m}$ of natural numbers such that $\mathfrak{x}$ is $\mathsf{A}_0(\mathfrak{m},x)$-convergent.

(iii) The sequence $\mathfrak{x}$ is A_0-*convergent* if there is a sequence $\mathfrak{m}$ of natural numbers and a $y\in\mathbb{R}$ such that $\mathfrak{x}$ is $\mathsf{A}_0(\mathfrak{m},y)$-convergent.

(iv) The sequence $\mathfrak{x}$ is $\mathsf{A}(x)$-*convergent* if it is $\mathsf{A}_0(\mathfrak{m},x)$-convergent for every natural number sequence $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ that satisfies

$$\lim_{n\to\infty} m_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{m_n}{n} = 0\,. \tag{2.15}$$

If so, we use the notation

$$\operatorname*{alim}_{i\to\infty} x_i = x\,.$$

(v) The sequence $\mathfrak{x}$ is A-convergent if there exists a $y \in \mathbb{R}$ such that $\mathfrak{x}$ is $\mathsf{A}(y)$-convergent.

Lemma 2.12. *Let $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ be a sequence of natural numbers, and let $r,x,y\in\mathbb{R}$. If $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ is $\mathsf{A}_0(\mathfrak{m},x)$-convergent and $\mathfrak{y} := \{y_i\}_{i\in\mathbb{N}}$ is $\mathsf{A}_0(\mathfrak{m},y)$-convergent, then $\mathfrak{x}+r\mathfrak{y}$ is $\mathsf{A}_0(\mathfrak{m},x+ry)$-convergent.*

Proof. This follows immediately from the triangle inequality. □

Lemma 2.13. *Let $J \subset \mathbb{N}$ be an infinite set with the property that there exists a positive integer sequence $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} m_n = \infty \tag{2.16}$$

and a $\delta \in [0,1)$ *such that*

$$\lim_{n\to\infty} \max_{0\le j\le \lfloor n/m_n\rfloor} \left| \frac{k_n(j,m_n)}{m_n} - \delta \right| = 0\,, \tag{2.17}$$

where

$$k_n(j,m_n) := \left|\{1 \le i \le m_n : jm_n + i \in \mathbb{N}\setminus J\}\right|.$$

Let $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ *be a real-valued sequence whose restriction to* J *converges to some* $x \in \mathbb{R}$*, and whose restriction to* $\mathbb{N}\setminus J$ *is* 0*. Then* $\mathfrak{x}$ *is* $\mathsf{A}_0(\mathfrak{m},(1-\delta)x)$*-convergent.*

Proof. Because $x_i = 0$ for all $i \in \mathbb{N}\setminus J$, we have

$$\begin{aligned}\tilde{x}_n(m_n,(1-\delta)x) &= \max_{0\le j\le\lfloor n/m_n\rfloor} \left| \frac{1}{m_n} \sum_{\substack{i=1\\ jm_n+i\in J}}^{m_n} (x_{jm_n+i} - x) - \frac{k_n(j,m_n)}{m_n}x + \delta x \right| \\ &\le \underbrace{\max_{0\le j\le\lfloor n/m_n\rfloor} \left| \frac{1}{m_n} \sum_{\substack{i=1\\ jm_n+i\in J}}^{m_n} (x_{jm_n+i} - x) \right|}_{U_1(n)} \\ &\quad + \underbrace{x \max_{0\le j\le\lfloor n/m_n\rfloor} \left| \frac{k_n(j,m_n)}{m_n} - \delta \right|}_{U_2(n)}\end{aligned}$$

for all $n \in \mathbb{N}$. Assumption (2.17) immediately yields $U_2(n) \to 0$. To show that $U_1(n) \to 0$ as well, we choose an arbitrary $\varepsilon > 0$. Since $\mathfrak{x}$ converges to x on J, there is an $N_0(\varepsilon) \in \mathbb{N}$ such that

$$|x_i - x| < \frac{\varepsilon}{3} \qquad \text{for all } i \in J,\ i \ge N_0(\varepsilon).$$

Because of (2.16), there exists an $N_1(\varepsilon) \in \mathbb{N}$ such that

$$m_n > N_0(\varepsilon) \quad \text{and} \quad \frac{1}{m_n}\sum_{i=1}^{N_0(\varepsilon)} |x_i - x| < \frac{\varepsilon}{3} \qquad \text{for all } n \ge N_1(\varepsilon).$$

It follows for all $n \geq N_1(\varepsilon)$ that

$$
\begin{aligned}
U_1(n) &= \max_{0 \leq j \leq \lfloor n/m_n \rfloor} \left| \frac{1}{m_n} \sum_{\substack{i=1 \\ jm_n+i \in J}}^{m_n} (x_{jm_n+i} - x) \right| \\
&\leq \max_{0 \leq j \leq \lfloor n/m_n \rfloor} \frac{1}{m_n} \sum_{\substack{i=1 \\ jm_n+i \in J}}^{m_n} |x_{jm_n+i} - x| \\
&\leq \frac{1}{m_n} \sum_{\substack{i=1 \\ i \in J}}^{N_0(\varepsilon)} |x_i - x| + \frac{1}{m_n} \sum_{\substack{i=N_0(\varepsilon)+1 \\ i \in J}}^{m_n} |x_i - x| \\
&\quad + \max_{1 \leq j \leq \lfloor n/m_n \rfloor} \frac{1}{m_n} \sum_{\substack{i=1 \\ jm_n+i \in J}}^{m_n} |x_{jm_n+i} - x| \\
&< \frac{\varepsilon}{3} + \frac{m_n - N_0(\varepsilon)}{m_n} \frac{\varepsilon}{3} + \max_{1 \leq j \leq \lfloor n/m_n \rfloor} \frac{m_n - k_n(j, m_n)}{m_n} \frac{\varepsilon}{3} \\
&\leq \varepsilon \,.
\end{aligned}
$$

This proves the lemma. $\square$

Corollary 2.14. *If a real-valued sequence $\mathfrak{x}$ converges to $x \in \mathbb{R}$, then $\mathfrak{x} := \{x_i\}$ is $\mathsf{A}_0(\mathfrak{m}, x)$-convergent for every positive integer sequence $\mathfrak{m} := \{m_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} m_n = \infty$. In particular, every convergent sequence is A-convergent.*

Moreover, if, for some constant $C > 0$ and $0 < \delta < 1$

$$
|x_i - x| \leq C \frac{1}{i^\delta} \qquad \text{for all } i \in \mathbb{N},
$$

then

$$
\tilde{x}_n(m_n, x) \leq \frac{C}{1-\delta} \frac{1}{m_n^\delta} \qquad \text{for all } i \in \mathbb{N}.
$$

Proof. We apply Lemma 2.13 with $J := \mathbb{N}$. Then (2.16) and (2.17) are satisfied with $\delta := 0$ for every sequence $m_n \to \infty$. The second part of the lemma follows with standard integral estimates. $\square$

Lemma 2.15. *Let $\mathfrak{m} := \{m_n\}_{n \in \mathbb{N}}$ be a sequence of natural numbers such that $\lim_{n \to \infty} m_n / n = 0$. Let $x \in \mathbb{R}$, and let $\mathfrak{x} := \{x_i\}_{i \in \mathbb{N}}$ be $\mathsf{A}_0(\mathfrak{m}, x)$-convergent. Assume that one of the following two conditions is satisfied:*

(i) The sequences $\mathfrak{x}$ and $\mathfrak{m}$ satisfy

$$\lim_{n\to\infty} \frac{m_n}{n} \sup_{\lfloor n/m_n \rfloor m_n < i \le n} |x_i - x| = 0\,.$$

(ii) The sequence m_n grows monotonically, and for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $m_n = k$.
Then $\mathfrak{x}$ converges in the sense of Cesàro towards x, that is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = x\,.$$

In particular, every A-convergent sequence is Cesàro convergent.

Proof. By assumption, $b_n := \lfloor n/m_n \rfloor \to \infty$ as $n \to \infty$. Let n be large enough for $b_n \ge 1$. Under condition (i) we consider

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right| &\le \frac{1}{b_n}\sum_{j=0}^{b_n-1}\frac{1}{m_n}\left|\sum_{i=1}^{m_n}\big(x_{jm_n+i} - x\big)\right| + \frac{1}{n}\sum_{i=b_n m_n+1}^{n}|x_i - x| \\ &\le \tilde{x}_n(m_n, x) + \frac{m_n}{n}\sup_{b_n m_n < i \le n}|x_i - x|\,,\end{aligned}$$

and the Cesàro convergence is immediate. If (ii) holds true, we conclude that, because $m_n \le n$ for every n large enough,

$$\left|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right| \le \sup_{k \ge m_n}\left|\frac{1}{k}\sum_{i=1}^{k} x_i - x\right| \xrightarrow{n\to\infty} 0\,. \qquad \square$$

Lemma 2.16. *Let $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ be a sequence that converges to x in the sense of Cesàro. Then $\mathfrak{x}$ is $\mathsf{A}_0(\mathfrak{m}, x)$-convergent for every positive integer sequence $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ which satisfies $\lim_{n\to\infty} m_n = \infty$ and*

$$\lim_{n\to\infty} \frac{n}{m_n} \sup_{k \ge m_n}\left|\frac{1}{k}\sum_{i=1}^{k} x_i - x\right| = 0\,.$$

Proof. We define

$$j_n := \operatorname*{arg\,max}_{0 \le j \le \lfloor n/m_n \rfloor} \frac{1}{m_n}\left|\sum_{i=1}^{m_n}(x_{jm_n+i} - x)\right| \qquad \text{for all } n \in \mathbb{N}.$$

Then it follows that

$$
\begin{aligned}
\tilde{x}_n(m_n, x) &\le \frac{1}{m_n}\left|\sum_{i=1}^{m_n}(x_{j_n m_n + i} - x)\right| \\
&\le \frac{1}{m_n}\left|\sum_{i=1}^{(j_n+1)m_n}(x_i - x)\right| + \frac{1}{m_n}\left|\sum_{i=1}^{j_n m_n}(x_i - x)\right| \\
&\le (2j_n + 1)\sup_{k\ge m_n}\left|\frac{1}{k}\sum_{i=1}^{k} x_i - x\right|.
\end{aligned}
$$

This proves the lemma. □

Example 2.17. If the sequence $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ goes to infinity too slowly, then Cesàro convergence of $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ does not necessarily entail $\mathsf{A}_0(\mathfrak{m}, x)$-convergence of $\mathfrak{x}$, even if the sequence $\mathfrak{x}$ is non-negative and bounded. An example is given by choosing

$$m_n := \lfloor \log n \rfloor \vee 1 \qquad \text{for all } n \in \mathbb{N},$$

and

$$x_i := 1 + \sin\big((\log i)^2\big) \qquad \text{for all } i \in \mathbb{N}.$$

Then $\mathfrak{x}$ converges to 1 in the sense of Cesàro, but

$$\frac{1}{m_n}\sum_{i=\lfloor n/m_n\rfloor m_n+1}^{(\lfloor n/m_n\rfloor+1)m_n} x_i$$

oscillates as $n \to \infty$. In particular, $\mathfrak{x}$ is not $\mathsf{A}_0(\mathfrak{m}, x)$-convergent for any $x \in \mathbb{R}$.

Lemma 2.18. *Every non-negative* A*-convergent sequence is bounded.*

Proof. Assume that $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ is a non-negative unbounded sequence. This implies that $\max_{1\le i\le n} x_i \to \infty$. We define

$$m_n := \left\lfloor \Big(n \wedge \max_{1\le i\le n} x_i\Big)^{1/2} \right\rfloor \vee 1 \qquad \text{for all } n \in \mathbb{N}.$$

Then $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ is a sequence of natural numbers that satisfies (2.15).

For every $n \in \mathbb{N}$ there are unique non-negative integers i_n^* and j_n^* such that $0 \le j_n^* \le \lfloor n/m_n \rfloor$, $0 \le i_n^* < m_n$ with the property that $\max_{1\le i\le n} x_i =$

$x_{j_n^* m_n + i_n^*}$. Let $x \in \mathbb{R}$ be chosen arbitrarily. Since $\mathfrak{x}$ is non-negative, it follows that

$$\tilde{x}_n(m_n, x) \geq \frac{1}{m_n} \sum_{i=1}^{m_n} x_{j_n^* m_n + i} - x \geq \frac{x_{j_n^* m_n + i_n^*}}{m_n} - x \qquad \text{for all } n \in \mathbb{N}.$$

By construction, the right hand side converges to ∞ as $n \to \infty$, and so does $\tilde{x}_n(m_n, x)$. Therefore, $\mathfrak{x}$ is not $\mathsf{A}(x)$-convergent for any $x \in \mathbb{R}$. In other words, $\mathfrak{x}$ is not A-convergent. □

Lemma 2.19. *Let $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ be a sequence of natural numbers such that $\lim_{n\to\infty} m_n/n = 0$, and let $x \in \mathbb{R}$. Let $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ be a bounded $\mathsf{A}_0(\mathfrak{m}, x)$-convergent sequence. Then*

$$\ell(n) := \exp\left(\sum_{i=1}^{n} \frac{x_i - x}{i}\right), \qquad n \in \mathbb{N},$$

is slowly varying at infinity.

Proof. Theorem A.3 implies that it is enough to show that

$$\lim_{n\to\infty} \frac{\ell(\lfloor \lambda n \rfloor)}{\ell(n)} = 1 \qquad \text{for all } 0 < \lambda < 1.$$

We fix any such λ, and note that

$$\frac{\ell(\lfloor \lambda n \rfloor)}{\ell(n)} = \exp\left(- \sum_{i=\lfloor \lambda n \rfloor + 1}^{n} \frac{x_i - x}{i}\right) \qquad \text{for all } n \in \mathbb{N}.$$

Our assumption $m_n = \mathrm{o}(n)$ implies $\lfloor \lambda n \rfloor / m_n \to \infty$ as $n \to \infty$. Thus, for n large enough we may apply Lemma A.15 (iii) with $l = l_n := \lfloor \lambda n \rfloor$ and $m = m_n$. Since by assumption $\mathfrak{x}$ is $\mathsf{A}_0(\mathfrak{m}, x)$-convergent with $x'_{\sup} := |x| \vee \sup_{i\in\mathbb{N}} |x_i| < \infty$, the right hand side of (A.19) converges to 0 as $n \to \infty$. This yields

$$\lim_{n\to\infty} \exp\left(- \sum_{i=\lfloor \lambda n \rfloor + 1}^{n} \frac{x_i - x}{i}\right) = 1\,. \qquad \square$$

2.2.2 Integer skeletons of periodic functions

Lemma 2.20. *Let $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ be the integer skeleton of a real-valued function with rational period. Let $q \in \mathbb{N}$ be the smallest integer multiple of the period, and let*

$$x := \frac{1}{q} \sum_{i=1}^{q} x_i\,. \tag{2.18}$$

Then $\mathfrak{x}$ *is* $\mathsf{A}_0(\mathfrak{m}, x)$*-convergent for every positive integer sequence* $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} m_n = \infty$*. More precisely, we have*

$$\tilde{x}_n(m_n, x) \leq \frac{2qx_{\sup}}{m_n} \qquad \textit{for all } n \in \mathbb{N},$$

where $x_{\sup} := \sup_{i\in\mathbb{N}}|x_i|$*. In particular, if* m_n *is a multiple of* q *for every* $n \in \mathbb{N}$*, we have* $\tilde{x}_n(m_n, x) = 0$ *for all* $n \in \mathbb{N}$*.*

Proof. Let $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers such that $m_n \to \infty$, and let x be as in (2.18). By construction,

$$x = \frac{1}{q}\sum_{i=1}^{q} x_{k+i} \qquad \text{for all } k \in \mathbb{Z}_+,$$

and thus

$$\sum_{i=1}^{\lfloor m_n/q\rfloor q} (x_{jm_n+i} - x) = 0 \qquad \text{for all } j \in \mathbb{Z}_+.$$

This leads to

$$\begin{aligned} m_n\tilde{x}_n(m_n, x) &= \sup_{0\leq j\leq\lfloor n/m_n\rfloor}\left|\sum_{i=1}^{\lfloor m_n/q\rfloor q}(x_{jm_n+i} - x) + \sum_{i=\lfloor m_n/q\rfloor q+1}^{m_n}(x_{jm_n+i} - x)\right| \\ &\leq 2qx_{\sup}\,, \end{aligned}$$

for all $n \in \mathbb{N}$, which proves the lemma. □

The situation of integer skeletons of function with irrational period is more elaborate. Here, our arguments are based on results from the theory of sequences uniformly distributed modulo 1. Convergence rates depend on the irrationality type of the frequency of the function. We refer to Appendix A.1.5 for a brief account of this theory.

Lemma 2.21. *Let* $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ *be the integer skeleton of a real-valued function* $x(t)$*,* $t \in \mathbb{R}$*, with irrational period* p *and bounded variation on any closed interval. Let*

$$x := \frac{1}{p}\int_0^p x(t)\,dt\,. \tag{2.19}$$

Then $\mathfrak{x}$ *is* $\mathsf{A}_0(\mathfrak{m}, x)$*-convergent for every positive integer sequence* $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} m_n = \infty$*.*

If the frequency $1/p$ is of finite type $1 \le \eta < \infty$, we have

$$\tilde{x}_n(m_n, x) = \mathrm{O}\Big(\frac{1}{m_n^\delta}\Big) \qquad \text{for every } 0 < \delta < 1/\eta. \tag{2.20}$$

If $1/p$ is of type $\eta = 1$, we even have

$$\tilde{x}_n(m_n, x) = \mathrm{O}\Big(\frac{(\log m_n)^2}{m_n}\Big). \tag{2.21}$$

Proof. Throughout the proof we use i exclusively to denote $\sqrt{-1}$. For each $n \in \mathbb{N}$ and $0 \le j \le \lfloor n/m_n \rfloor$, we define

$$t_k^{(j,m_n)} := \frac{jm_n}{p} + \frac{k}{p} \qquad \text{for all } 1 \le k \le m_n.$$

Let

$$D_{m_n}^*(j) := \sup_{0 \le \alpha < 1} \left| \frac{1}{m_n} \left| \{1 \le k \le m_n : t_k^{(j,m_n)} - \lfloor t_k^{(j,m_n)} \rfloor \le \alpha\} \right| - \alpha \right|$$

be the discrepancy of the finite sequence $t_1^{(j,m_n)}, \dots, t_{m_n}^{(j,m_n)}$. We define a function $f(t) := x(pt)$, $t \in \mathbb{R}$, which, by definition, has period 1 and bounded variation, $V(f)$ say, on $[0,1]$. It follows that

$$\begin{aligned}
\tilde{x}_n(m_n, x) &= \max_{0 \le j \le \lfloor n/m_n \rfloor} \left| \frac{1}{m_n} \sum_{k=1}^{m_n} x_{jm_n+k} - x \right| \\
&= \max_{0 \le j \le \lfloor n/m_n \rfloor} \left| \frac{1}{m_n} \sum_{k=1}^{m_n} x\big(pt_k^{(j,m_n)}\big) - \frac{1}{p} \int_0^p x(t)\, dt \right| \\
&= \max_{0 \le j \le \lfloor n/m_n \rfloor} \left| \frac{1}{m_n} \sum_{k=1}^{m_n} f\big(t_k^{(j,m_n)}\big) - \int_0^1 f(t)\, dt \right| \\
&= \max_{0 \le j \le \lfloor n/m_n \rfloor} \left| \frac{1}{m_n} \sum_{k=1}^{m_n} f\big(t_k^{(j,m_n)} - \lfloor t_k^{(j,m_n)} \rfloor\big) - \int_0^1 f(t)\, dt \right|,
\end{aligned}$$

and the Koksma inequality (Theorem A.10) implies that

$$\tilde{x}_n(m_n, x) \le V(f) \max_{0 \le j \le \lfloor n/m_n \rfloor} D_{m_n}^*(j) \qquad \text{for all } n \in \mathbb{N}. \tag{2.22}$$

The Erdős-Turán theorem (Theorem A.9) yields

$$\max_{0 \le j \le \lfloor n/m_n \rfloor} D_{m_n}^*(j) \le \frac{6}{r+1} + \frac{4}{\pi} \sum_{l=1}^{r} \frac{1}{l} \left| \frac{1}{m_n} \sum_{k=1}^{m_n} \mathrm{e}^{2\pi i l k/p} \right| \qquad \text{for all } r \in \mathbb{N}. \tag{2.23}$$

Since p is irrational, the sequence $\{k/p\}_{r\in\mathbb{N}}$ is uniformly distributed modulo 1 (Example A.7), and therefore the Weyl criterion (Theorem A.6) and $m_n \to \infty$ imply

$$\lim_{n\to\infty} \frac{1}{m_n} \sum_{k=1}^{m_n} \mathrm{e}^{2\pi i l k/p} = 0 \qquad \text{for all } l \in \mathbb{Z}\setminus\{0\}.$$

We obtain

$$\lim_{n\to\infty} \max_{0\le j\le \lfloor n/m_n\rfloor} D^*_{m_n}(j) \le \frac{6}{r+1} \qquad \text{for all } r \in \mathbb{N},$$

and hence

$$\lim_{n\to\infty} \max_{0\le j\le \lfloor n/m_n\rfloor} D^*_{m_n}(j) = 0\,. \tag{2.24}$$

The first part of the lemma follows from (2.22) and (2.24).

To prove the remaining assertions, we use $\langle t\rangle$ to denote the distance of a real number t from its nearest integer. Note that (cf. Kuipers and Niederreiter (1974, Chapter 2, Proof of Lemma 3.2))

$$\left|\sum_{k=1}^{m_n} \mathrm{e}^{2\pi i l k/p}\right| \le \frac{1}{|\sin(\pi l/p)|} \le \frac{1}{2\langle l/p\rangle} \qquad \text{for all } l, n \in \mathbb{N}.$$

Then (2.23) implies

$$\max_{0\le j\le \lfloor n/m_n\rfloor} D^*_{m_n}(j) \le \frac{6}{r+1} + \frac{2}{\pi}\frac{1}{m_n}\sum_{l=1}^{r}\frac{1}{l\langle l/p\rangle} \qquad \text{for all } r \in \mathbb{N}.$$

Now let $1/p$ be of finite type η. By Lemma A.12, for every $\varepsilon > 0$ there is a constant $c(\eta,\varepsilon) > 0$ such that

$$\max_{0\le j\le \lfloor n/m_n\rfloor} D^*_{m_n}(j) \le \frac{6}{r+1} + \frac{2c(\eta,\varepsilon)}{\pi}\frac{1}{m_n} r^{\eta-1+\varepsilon} \qquad \text{for all } r \in \mathbb{N}.$$

Set $r = r_n := \lfloor m_n^{1/\eta}\rfloor$. Then

$$\max_{0\le j\le \lfloor n/m_n\rfloor} D^*_{m_n}(j) = \mathrm{O}\left(\frac{1}{m_n^{(1-\varepsilon)/\eta}}\right),$$

and (2.20) follows from (2.22).

If $1/p$ is of type $\eta = 1$, then Lemma A.12 implies that, for some constant $c > 0$,

$$\max_{0\le j\le \lfloor n/m_n\rfloor} D^*_{m_n}(j) \le \frac{6}{r+1} + \frac{2c}{\pi}\frac{1}{m_n}(\log r)^2 \qquad \text{for all } r \in \mathbb{N}.$$

Now we choose $r = r_n := m_n$, and we obtain

$$\max_{0\le j\le\lfloor n/m_n\rfloor} D^*_{m_n}(j) = \mathrm{O}\biggl(\frac{(\log m_n)^2}{m_n}\biggr)\,,$$

and conclude (2.21) from (2.22). $\square$

Lemma 2.22. *Let* $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ *be the integer skeleton of a sinusoidal function*

$$x(t) := x + \sum_{l=1}^{N} \lambda_l \cos(2\pi f_l t - \varphi_l)\,, \qquad t\in\mathbb{R},$$

with amplitudes $\lambda_l > 0$*, frequencies* $f_l > 0$ *and phases* $0 \le \varphi_l < 2\pi$*, for* $l = 1,\dots,N$*. Then* $\mathfrak{x}$ *is* $\mathsf{A}_0(\mathfrak{m}, x)$*-convergent for every positive integer sequence* $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} m_n = \infty$*. More precisely, we have*

$$\tilde{x}_n(m_n, x) = \mathrm{O}\Bigl(\frac{1}{m_n}\Bigr)\,.$$

Proof. The period of the l-th component $\lambda_l\cos(2\pi f_l t - \varphi_l)$ of $x(t)$ is $p_l := 1/f_l$. Without loss of generality we may assume that the periods $p_1,\dots,p_{N_0}$ of the first N_0 components are rational, and the periods $p_{N_0+1},\dots,p_N$ of the remaining $N - N_0$ components are irrational. Then we have

$$\begin{aligned}
\tilde{x}_n(m_n, x) &= \sup_{0\le j\le\lfloor n/m_n\rfloor}\biggl|\frac{1}{m_n}\sum_{k=1}^{m_n}\sum_{l=1}^{N}\lambda_l\cos\bigl(2\pi f_l(jm_n+k)-\varphi_l\bigr)\biggr| \\
&\le \sum_{l=1}^{N_0}\underbrace{\sup_{0\le j\le\lfloor n/m_n\rfloor}\biggl|\frac{1}{m_n}\sum_{k=1}^{m_n}\lambda_l\cos\bigl(2\pi f_l(jm_n+k)-\varphi_l\bigr)\biggr|}_{U_1(l,n)} \\
&\quad + \sum_{l=N_0+1}^{N}\underbrace{\sup_{0\le j\le\lfloor n/m_n\rfloor}\biggl|\frac{1}{m_n}\sum_{k=1}^{m_n}\lambda_l\cos\bigl(2\pi f_l(jm_n+k)-\varphi_l\bigr)\biggr|}_{U_2(l,n)}
\end{aligned}$$

Since $\lambda_l\cos(2\pi f_l t - \varphi_l)$, $t\in\mathbb{R}$, is a function with rational period p_l for $1\le l\le N_0$, we invoke Lemma 2.20, and we obtain

$$U_1(l,n) \le \frac{2q_l\lambda_l}{m_n} \qquad \text{for all } n\in\mathbb{N} \text{ and } 1\le l\le N_0,$$

where $q_l\in\mathbb{N}$ is the smallest integer multiple of the period.

If $N_0 < l \le N$, that is, if f_l is irrational, then we have for any such l and every $n \in \mathbb{N}$

$$\begin{aligned}
U_2(l,n) &= \sup_{0\le j\le \lfloor n/m_n\rfloor} \frac{\lambda_l}{2m_n}\left|\sum_{k=1}^{m_n}\left(\mathrm{e}^{2\pi i f_l(jm_n+k)-i\varphi_l} + \mathrm{e}^{-2\pi i f_l(jm_n+k)+i\varphi_l}\right)\right| \\
&\le \frac{\lambda_l}{2m_n}\left(\left|\sum_{k=1}^{m_n}\mathrm{e}^{2\pi i f_l k}\right| + \left|\sum_{k=1}^{m_n}\mathrm{e}^{-2\pi i f_l k}\right|\right) \\
&\le \frac{\lambda_l}{2m_n}\left(\frac{1}{|\sin(\pi f_l)|} + \frac{1}{|\sin(-\pi f_l)|}\right) \\
&= \frac{\lambda_l}{|\sin(\pi f_l)|}\frac{1}{m_n}\,.
\end{aligned}$$

Thus,

$$\tilde{x}_n(m_n,x) \le \left(2\sum_{l=1}^{N_0} q_l\lambda_l + \sum_{l=N_0+1}^{N}\frac{\lambda_l}{|\sin(\pi f_l)|}\right)\frac{1}{m_n} \qquad \text{for all } n\in\mathbb{N}. \qquad \square$$

2.3 The quasi-logarithmic condition

After having introduced the notion of A-convergence, we return to the task of weakening the logarithmic condition LC from Subsection 2.1.3, by replacing convergence with A-convergence in Definition 2.8.

2.3.1 Condition UQLC

Definition 2.23. A sequence $\{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables satisfies the *quasi-logarithmic condition* $\mathsf{QLC}_0(\theta,\mathfrak{m})$ for a constant $\theta > 0$ and a sequence $\mathfrak{m} := \{m_n\}_{n\in\mathbb{N}}$ of natural numbers, if the sequence $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ is $\mathsf{A}_0(\theta,\mathfrak{m})$-convergent and if

$$\lim_{i\to\infty} i(\mathbb{E}Z_i - \mathbb{P}[Z_i = 1]) = 0\,. \tag{2.25}$$

We say that $\{Z_i\}_{i\in\mathbb{N}}$ satisfies the *quasi-logarithmic condition* $\mathsf{QLC}(\theta)$ if

$$\operatorname*{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta \tag{2.26}$$

(that is, if $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ is $\mathsf{A}(\theta)$-convergent), and if (2.25) holds true.

Conditions (2.26) and (2.25) are counterparts of the conditions (2.11) and (2.13) of the logarithmic condition $\mathsf{LC}(\theta)$ in Definition 2.8. Note that under condition $\mathsf{QLC}(\theta)$, from Lemma 2.18,

$$\sup_{i\in\mathbb{N}} i\mathbb{E}Z_i < \infty\,, \qquad \text{or, equivalently,} \qquad \mathbb{E}Z_i = \mathrm{O}(1/i)\,.$$

Lemma 2.24. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of $\mathbb{Z}_+$-valued random variables. Then we have for any constant $\theta > 0$*

$$\{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{LC}(\theta) \quad \Rightarrow \quad \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{QLC}(\theta).$$

Proof. In view of Corollary 2.14, (2.11) implies (2.26). Moreover, under $\mathsf{LC}(\theta)$, (2.12) is equivalent to (2.13), that is, to (2.25). □

We now combine the quasi-logarithmic condition with the uniformity condition UC from Subsection 2.1.1.

Definition 2.25. A sequence $\{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables satisfies the *uniform quasi-logarithmic condition* $\mathsf{UQLC}(\theta, \mathfrak{r})$ for a constant $\theta > 0$ and a positive integer sequence $\mathfrak{r}$ if it satisfies $\mathsf{QLC}(\theta)$ and the uniformity condition $\mathsf{UC}(\mathfrak{r})$.

Lemma 2.26. *For any $\theta > 0$ and any positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$*

$$\{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{UQLC}(\theta,\mathfrak{r}) \qquad \Leftrightarrow \qquad \begin{array}{l} \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{UC}(\mathfrak{r}), \\ \textit{and } \operatorname{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta. \end{array}$$

Proof. We only have to show the implication from the right to the left hand side. First, condition $\mathsf{UC}(\mathfrak{r})$ yields $\varepsilon_{i1}(i\mathbb{E}Z_i, r_i) \to 0$. Second, we conclude from Lemma 2.18 that the sequence $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ is bounded, and hence that $\mathbb{E}Z_i \to 0$. Now, (A.9) implies (2.25) □

2.3.2 Condition UQLC for standard distributions on $\mathbb{Z}_+$

If the $\{Z_i\}_{i\in\mathbb{N}}$ are all either Poisson, binomially or the negative binomially distributed, then $\mathsf{UQLC}(\theta, \mathfrak{r})$ follows from (2.26) already.

Lemma 2.27. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent Poisson distributed random variables. For any constant $\theta > 0$ and any positive integer sequence $\mathfrak{r}$ we have*

$$\operatorname*{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta \qquad \Leftrightarrow \qquad \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{UQLC}(\theta,\mathfrak{r}). \tag{2.27}$$

In this case we have

$$\mu_i(\mathfrak{r}) = \mathrm{O}(1/i)\,. \tag{2.28}$$

Proof. If the sequence $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ is $\mathsf{A}(\theta)$-convergent, it is bounded (cf. Lemma 2.18), and thus $\mathbb{E}Z_i = \mathrm{O}(1/i)$. Then Lemma 2.5 yields (2.28), and $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{UC}(\mathfrak{r})$. The lemma follows from Lemma 2.26. □

Lemma 2.28. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent binomially distributed random variables, $Z_i \sim \mathrm{Bin}(r_i, p_i)$ for all $i \in \mathbb{N}$, where $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ is a sequence of natural numbers and $0 \le p_i \le 1$ for all $i \in \mathbb{N}$. For any constant $\theta > 0$ we have*

$$\operatorname*{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta \quad \Leftrightarrow \quad \operatorname*{alim}_{i\to\infty} ip_ir_i = \theta \quad \Leftrightarrow \quad \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{UQLC}(\theta, \mathfrak{r}). \tag{2.29}$$

If so,

$$\mu_i(\mathfrak{r}) = 0 \qquad \textit{for all } i \in \mathbb{N}. \tag{2.30}$$

Proof. The first equivalence is obvious, because $\mathbb{E}Z_i = r_ip_i$ for all $i \in \mathbb{N}$. Equation (2.30) holds true because of Lemma 2.6. But this means that $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{UC}(\mathfrak{r})$. Now invoke Lemma 2.26. □

Lemma 2.29. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent negative binomially distributed random variables, $Z_i \sim \mathrm{NB}(r_i, p_i)$ for all $i \in \mathbb{N}$, where $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ is a sequence of natural numbers and $0 \le p_i < 1$ for all $i \in \mathbb{N}$. For any constant $\theta > 0$ we have*

$$\operatorname*{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta \quad \Leftrightarrow \quad \operatorname*{alim}_{i\to\infty} ip_ir_i = \theta \quad \Leftrightarrow \quad \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{UQLC}(\theta, \mathfrak{r}). \tag{2.31}$$

If so,

$$\mu_i(\mathfrak{r}) = \mathrm{O}(1/i)\,. \tag{2.32}$$

Proof. We have

$$i\mathbb{E}Z_i = ir_ip_i + p_i i\mathbb{E}Z_i = ir_ip_i + \frac{p_i}{1-p_i} ir_ip_i \qquad \text{for all } i \in \mathbb{N}. \tag{2.33}$$

If $\operatorname{alim}_{i\to\infty} ir_ip_i = \theta$, then $\{ir_ip_i\}_{i\in\mathbb{N}}$ is bounded and $p_i \to 0$. Thus (2.33) yields $i\mathbb{E}Z_i = ir_ip_i + \mathrm{o}(1)$. Lemma 2.12 and Corollary 2.14 imply $\operatorname{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta$. If $\operatorname{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta$, then $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ is bounded and $p_i \to 0$. Hence, $ir_ip_i = i\mathbb{E}Z_i + \mathrm{o}(1)$, and (2.31) follows by invoking Lemma 2.12 and Corollary 2.14 once more.

The second equivalence in (2.31) is a consequence of Lemma 2.7 and Lemma 2.26. Equation (2.32) follows from Lemma 2.7 as well. □

Remark 2.30. If the A-convergence in the conditions (2.27), (2.29) or (2.31) is replaced by the usual notion of convergence, then the sequences $\{Z_i\}_{i\in\mathbb{N}}$ in Lemma 2.27, 2.28 and 2.29, respectively, satisfy condition $\mathsf{ULC}(\theta, \mathfrak{r})$.

2.3.3 Condition SUQLC

Unlike condition ULC, condition UQLC does not prevent $\mathbb{E}Z_i$ from being equal to 0 for infinitely many $i \in \mathbb{N}$. If, for example, $Z_i = 0$ for every odd i, sums of the form $\sum_{i=1}^n iZ_i$ are concentrated on the even integers, and a local Dickman approximation theorem as in (1.7) cannot hold. We therefore impose an additional contraint on condition UQLC in order to prevent such situations.

Definition 2.31. Let $\{Z_i\}_{i\in\mathbb{N}}$ be sequence of independent $\mathbb{Z}_+$-valued random variables.

(i) The sequence $\{Z_i\}_{i\in\mathbb{N}}$ satisfies the *smoothness condition* $\mathsf{SC}(\mathfrak{b})$ for a sequence $\mathfrak{b} := \{b_n\}_{n\in\mathbb{N}}$ of non-negative integers if it satisfies

$$\lim_{n\to\infty} d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum\nolimits_{i=b_n+1}^n iZ_i\Big), \mathscr{L}\Big(\sum\nolimits_{i=b_n+1}^n iZ_i + 1\Big)\Big) = 0\,.$$

(ii) The sequence $\{Z_i\}_{i\in\mathbb{N}}$ satisfies the *smoothness condition* SC if it satisfies condition $\mathsf{SC}(\mathfrak{b})$ for every non-negative integer sequence $\mathfrak{b} := \{b_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{b_n}{n} = 0\,. \tag{2.34}$$

Definition 2.32. A sequence $\{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables satisfies the *smoothed uniform quasi-logarithmic condition* $\mathsf{SUQLC}(\theta, \mathfrak{r}, \mathfrak{b})$ for a constant $\theta > 0$, a positive integer sequence $\mathfrak{r}$ and a non-negative integer sequence $\mathfrak{b} := \{b_n\}_{n\in\mathbb{N}}$ if it satisfies condition $\mathsf{SUQLC}(\theta, \mathfrak{r})$ and condition $\mathsf{SC}(\mathfrak{b})$.

The sequence $\{Z_i\}_{i\in\mathbb{N}}$ satisfies the *smoothed uniform quasi-logarithmic condition* $\mathsf{SUQLC}(\theta, \mathfrak{r})$ for a constant $\theta > 0$ and a positive integer sequence $\mathfrak{r}$ if it satisfies $\mathsf{UQLC}(\theta, \mathfrak{r}, \mathfrak{b})$ for every non-negative integer sequence $\mathfrak{b} := \{b_n\}_{n\in\mathbb{N}}$ as in (2.34).

This smoothed version of the UQLC-condition still is more general than condition UQLC^*, or equivalently, than condition UC^*.

Proposition 2.33. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables. Then we have for any $\theta > 0$ and any positive integer sequence $\mathfrak{r}$*

$$\{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{ULC}(\theta, \mathfrak{r}) \qquad \Rightarrow \qquad \{Z_i\}_{i\in\mathbb{N}} \textit{ satisfies } \mathsf{SUQLC}(\theta, \mathfrak{r})\,.$$

Proof. Lemma 2.10, Corollary 2.14 and Lemma 2.26 immediately imply that $\mathsf{ULC}(\theta, \mathfrak{r})$ entails $\mathsf{UQLC}(\theta, \mathfrak{r})$.

Under condition $\mathsf{ULC}(\theta, \mathfrak{r})$ we have $i\mathbb{P}[Z_i = 1] \to \theta$ (cf. equation (2.12)). What is more, (A.11) in Lemma A.13 implies that $\mathbb{P}[Z_i = 0] \to 1$, because

$\mathbb{E}Z_i \to 0$ and $\varepsilon_{i1}(i\mathbb{E}Z_i, r_i) \to 0$ under $\mathsf{UC}(\mathfrak{r})$. Hence, there is an $\varepsilon > 0$ and an $N_\varepsilon \in \mathbb{N}$ such that

$$\mathbb{P}[Z_i = 1] \geq \frac{\varepsilon}{i} \quad \text{and} \quad \mathbb{P}[Z_i = 0] \geq \varepsilon \qquad \text{for all } i \geq N_\varepsilon.$$

The sequence $Y_i := iZ_i$, $i \in \mathbb{N}$, satisfies Assumption 6.6 with $N = N_\varepsilon$, $\psi_0 = \psi_1 = \varepsilon$ and $g = 0$. We therefore can apply Theorem 6.18 for any non-negative integer sequence $a_n = \mathrm{o}(n)$, and conclude that $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition SC. □

The following lemma gives a simple criterion under which the smoothness condition SC is satisfied. It is based, as the previous proposition, on coupling results from Chapter 6. For refined conditions that are sufficient for condition SC, along with convergence rates for the total variation distance in Definition 2.31, we refer to Sections 6.3 and 6.4, and to Nietlispach (2007a).

Lemma 2.34. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables that satisfies the uniformity condition $\mathsf{UC}(\mathfrak{r})$. If*

$$0 < \inf\nolimits_{i\in\mathbb{N}} i\mathbb{E}Z_i < \sup\nolimits_{i\in\mathbb{N}} i\mathbb{E}Z_i < \infty\,, \tag{2.35}$$

then $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition SC.

Proof. The proof is similar to the proof of Proposition 2.33. □

The smoothness condition SC on $\{Z_i\}_{i\in\mathbb{N}}$ does not entail A-convergence of $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ and vice versa, as shows the following example.

Example 2.35. Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables that satisfies the uniformity condition $\mathsf{UC}(\mathfrak{r})$.

(i) Assume that $i\mathbb{E}Z_i = 2 + \sin((\log i)^2)$ for all $i \in \mathbb{N}$. Then $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ is not A-convergent (cf. Example 2.17), but (2.35) is satisfied. Therefore $\{Z_i\}_{i\in\mathbb{N}}$ satisfies the smoothness condition SC.

(ii) Now assume that $i\mathbb{E}Z_i = 0$ if i is odd and $i\mathbb{E}Z_i = 1$ if i is even. Thus, $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ is A-convergent by Lemma 2.20. But $\sum_{i=b+1}^n iZ_i$ is concentrated on the even integers for any $b \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, and the total variation distance between $\mathscr{L}(\sum_{i=b+1}^n iZ_i)$ and $\mathscr{L}(\sum_{i=b+1}^n iZ_i + 1)$ is equal to 1. Condition SC does not hold.

We take a look at some simple yet typical examples of quasi-logarithmic sequences $\{Z_i\}_{i\in\mathbb{N}}$.

Example 2.36. Assume that random variables $\{Z_i\}_{i\in\mathbb{N}}$ have standard distributions such as described in Lemma 2.27, 2.28 or 2.29. Assume that, for some constant $\theta > 0$,

$$\mathbb{E}Z_i = \frac{\theta}{i}(1+\cos i) \qquad \text{for all } i \in \mathbb{N}.$$

The sequence $\{iZ_i\}_{i\in\mathbb{N}}$ is $\mathsf{A}(\theta)$-convergent (cf. Lemma 2.22), and therefore satisfies condition $\mathsf{UQLC}(\theta, \mathfrak{r})$ for some positive integer sequence $\mathfrak{r}$, by invoking one of the Lemmas 2.27, 2.28 or 2.29. What is more,

$$\sum_{i=1}^{\infty} \frac{\mu_i(\mathfrak{r})}{i} < \infty\,.$$

Theorem 6.4 shows us that $\{Z_i\}_{i\in\mathbb{N}}$ even satisfies condition $\mathsf{SUQLC}(\theta, \mathfrak{r})$. However, the logarithmic condition $\mathsf{LC}(\theta)$ is not satisfied, because (2.11) does not hold.

Example 2.37. A similar example to the one above is given if the Z_i have standard distributions such as described in Lemma 2.27, 2.28 or 2.29, such that, for some constants $0 < \theta_1 < \theta_2$,

$$i\mathbb{E}Z_i = \begin{cases} \theta_1 & \text{for } i \text{ odd,} \\ \theta_2 & \text{for } i \text{ even,} \end{cases}$$

In fact, $\operatorname{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta := (\theta_1+\theta_2)/2$ holds because of Lemma 2.20, Corollary 2.14 and Lemma 2.12. Moreover, condition SC follows from Theorem 6.27.

Example 2.38. A similar example to the one above is given if the Z_i have standard distributions such as described in Lemma 2.27, 2.28 or 2.29, such that

$$i\mathbb{E}Z_i = \begin{cases} 1+\mathrm{o}(1) & \text{for } i \text{ odd,} \\ \mathrm{o}(1) & \text{for } i \text{ even,} \end{cases}$$

As in Example 2.36, the sequence $\{iZ_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{SUQLC}(\theta, \mathfrak{r})$ but not $\mathsf{LC}(\theta)$.

Example 2.39. Again, assume that random variables $\{Z_i\}_{i\in\mathbb{N}}$ have standard distributions such as described in Lemma 2.27, 2.28 or 2.29, such that

$$\mathbb{E}Z_i = \frac{\theta}{i}(1+\eta_i) \qquad \text{for all } i \in \mathbb{N},$$

for some constant $\theta > 0$ and a sequence $\{\eta_i\}_{i\in\mathbb{N}}$ that satisfies

$$\sum_{i=1}^{\infty} \frac{|\eta_i|}{i} = \infty\,. \tag{2.36}$$

Since $\lim_{i\to\infty} i\mathbb{E}Z_i = \theta$, it follows with Remark 2.30 that $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{ULC}(\theta,\mathfrak{r})$ (or by Lemma 2.10 equivalently $\mathsf{UC}^*(\theta,\mathfrak{r})$) for some positive integer sequence $\mathfrak{r}$, and that

$$\sum_{i=1}^{\infty} \frac{\mu_i(\mathfrak{r})}{i} < \infty\,.$$

However, (2.36) yields

$$\sum_{i=1}^{\infty} \frac{\mu_i(\theta,\mathfrak{r})}{i} = \infty\,,$$

since, as in the proof of Lemma 2.3, we have

$$\mu_i(\theta,\mathfrak{r}) \geq \sup_{j>i}\left|\frac{j\mathbb{E}Z_j}{\theta} - 1\right| \geq |\eta_{i+1}| \qquad \text{for all } i \in \mathbb{N}.$$

In particular, the working conditions of Arratia et al. (2000) and Arratia et al. (2003), described at the end of Subsection 2.1.1, are not satisfied in this situation. The sequence $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ does not converge quickly enough to θ.

Condition $\mathsf{SUQLC}(\theta,\mathfrak{r})$ is a *probabilistic* condition on the sequence $\{Z_i\}_{i\in\mathbb{N}}$ that directly generalizes the logarithmic conditions of Arratia et al. (2000, 2003), which are probabilistic conditions $\{Z_i\}_{i\in\mathbb{N}}$ on as well.

In the literature *analytic* conditions, similar to SUQLC, can be found. Arratia et al. (1995) and Stark (1997a, 1999) work with Poisson distributed random variables $\{Z_i\}_{i\in\mathbb{N}}$, imposing analytic conditions on the generation function

$$f(z) := \sum_{i=1}^{\infty} \mathbb{E}Z_i\, z^i\,.$$

Results similar to Theorem 5.19 and Corollary 5.20 are obtained. We refer to Subsubsection 5.2.2.1 for a further discussion.

3 Approximation by the Dickman distribution

3.1 Deviation from a compound Poisson distribution

3.1.1 The compound Poisson distribution

Definition 3.1. Let $\Lambda := \{\lambda_i\}_{i\in\mathbb{N}}$ be a non-negative sequence such that

$$\sum_{i=1}^{\infty} \lambda_i < \infty\,. \tag{3.1}$$

A $\mathbb{Z}_+$-valued random variable X has a *compound Poisson distribution* with rates λ_i, $i \in \mathbb{N}$, if the Laplace transform of X satisfies

$$\mathbb{E}\mathrm{e}^{-tX} = \exp\biggl(-\sum_{i=1}^{\infty}\bigl(1-\mathrm{e}^{-ti}\bigr)\lambda_i\biggr) \qquad \text{for all } t > 0. \tag{3.2}$$

We denote this distribution by $\mathrm{CP}(\Lambda)$.

It follows from (3.2) that if X has the compound Poisson distribution $\mathrm{CP}(\Lambda)$ then

$$\mathscr{L}(X) = \mathscr{L}\biggl(\sum_{i=1}^{\infty} iY_i\biggr)\,,$$

where $\{Y_i\}_{i\in\mathbb{N}}$ is a sequence of independent Poisson distributed random variables, $Y_i \sim \mathrm{Po}(\lambda_i)$ for all $i \in \mathbb{N}$. Also note that if $\lambda_i = 0$ for all $i \geq 2$, then the compound Poisson distribution $\mathrm{CP}(\Lambda)$ is the Poisson distribution $\mathrm{Po}(\lambda_1)$.

A useful characterization of the compound Poisson distribution is given in the following lemma.

Lemma 3.2 (Barbour et al. (1992, Corollary 1))**.** *A $\mathbb{Z}_+$-valued random variable X is compound Poisson distributed with rates λ_i, $i \in \mathbb{N}$, if and only if*

$$\mathbb{E}\{Xg(X)\} = \sum_{i=1}^{\infty} i\lambda_i \mathbb{E}g(X+i) \tag{3.3}$$

for all bounded functions $g : \mathbb{Z}_+ \to \mathbb{R}$.

Corollary 3.3. *Let* $\Lambda := \{\lambda_i\}_{i\in\mathbb{N}}$ *be a non-negative sequence such that condition* (3.1) *is satisfied. Assume that* X *is* $\mathrm{CP}(\Lambda)$*-distributed. It follows that*

$$k\mathbb{P}[X = k] = \sum_{i=1}^{\infty} i\lambda_i \mathbb{P}[X = k - i] \qquad \text{for all } k \in \mathbb{N}. \tag{3.4}$$

Proof. For each $k \in \mathbb{N}$ we consider the bounded function $g_k(j) := \mathbf{1}\{j = k\}$, $j \in \mathbb{Z}_+$. Now (3.4) follows from (3.3), setting $g := g_k$ for each $k \in \mathbb{N}$. □

Lemma 3.4 (Barbour et al. (1992, Theorem 3, Lemma 7))**.** *Let* $\Lambda := \{\lambda_i\}_{i\in\mathbb{N}}$ *be a non-negative sequence such that condition* (3.1) *is satisfied. Assume that* X *is* $\mathrm{CP}(\Lambda)$*-distributed. Then we have*

$$\mathbb{P}[X = k] = \exp\left(-\sum_{i=1}^{\infty} \lambda_i\right) \sum_{j=1}^{k} j\lambda_j a_{kj} \qquad \text{for all } k \in \mathbb{N},$$

where

$$a_{jj} = \frac{1}{j} \qquad \text{for all } j \in \mathbb{N}, \tag{3.5}$$

$$a_{j+l,j} = \frac{1}{j+l} \sum_{i=1}^{l} i\lambda_i a_{j+l-i,j} \qquad \text{for all } j, l \in \mathbb{N}. \tag{3.6}$$

We are interested in the following special compound Poisson distribution. Let $\theta > 0$ and $n \in \mathbb{N}$. Let $\Lambda := \{\lambda_i\}_{i\in\mathbb{N}}$ be the sequence defined by

$$\lambda_i := \begin{cases} \theta/i & \text{if } 1 \le i \le n, \\ 0 & \text{if } i > n. \end{cases}$$

We use $\mathrm{CP}(\theta, n) := \mathrm{CP}(\Lambda)$ to denote the compound Poisson distribution associated with this special sequence Λ.

3.1.2 Stein's method for the compound Poisson distribution

Stein's method for the compound Poisson distribution was developed by Barbor, Chen and Loh (1992). We give a brief outline of the main ideas. Let $\Lambda := \{\lambda_i\}_{i\in\mathbb{N}}$ be a non-negative sequence which has the property (3.1). Let

$$\mathcal{F} := \{f : \mathbb{Z}_+ \to \mathbb{R} : \mathbb{E}f(X) \text{ exists}\}.$$

For a function $f \in \mathcal{F}$ we consider the so-called Stein equation for $\mathrm{CP}(\Lambda)$ in the unknown $g : \mathbb{Z}_+ \to \mathbb{R}$, that is

$$f(k) - \mathbb{E}f(X) = \sum_{i=1}^{\infty} i\lambda_i g(k+i) - kg(k) \qquad \text{for all } k \in \mathbb{Z}_+. \tag{3.7}$$

Whether a bounded solution g_f of the Stein equation (3.7) exists (cf. Barbour et al. (1992, Section 3) and also Arratia et al. (2003, Section 9.1)), depends on properties of f and Λ.

Assume that the Stein equation (3.7) has a bounded solution g_f for every $f \in \mathcal{F}_0$, where $\mathcal{F}_0 \subset \mathcal{F}$ is some subset of $\mathcal{F}$. If so, we may conclude from (3.7) that

$$\mathbb{E}f(Y) - \mathbb{E}f(X) = \sum_{i=1}^{\infty} i\lambda_i \mathbb{E}g_f(Y+i) - \mathbb{E}\big\{Yg_f(Y)\big\} \qquad \text{for all } f \in \mathcal{F}_0,$$

for any $\mathbb{Z}_+$-valued random variable Y. This implies that

$$\begin{aligned} d_{\mathcal{F}_0}\big(\mathscr{L}(Y), \mathscr{L}(X)\big) &:= \sup\nolimits_{f \in \mathcal{F}_0} \big|\mathbb{E}f(Y) - \mathbb{E}f(X)\big| \\ &= \sup_{f \in \mathcal{F}_0} \left| \sum_{i=1}^{\infty} i\lambda_i \mathbb{E}g_f(Y+i) - \mathbb{E}\big\{Yg_f(Y)\big\} \right|. \end{aligned}$$

Note that $d_{\mathcal{F}_0}$ is a metric, associated with $\mathcal{F}_0$, on the set of probability measures on $\mathbb{Z}_+$. If $\mathscr{L}(Y)$ is close to $\mathscr{L}(X) = \mathrm{CP}(\Lambda)$, then, in view of the characterizing equation (3.3) of the $\mathrm{CP}(\Lambda)$-distribution, we may hope to show that

$$\left| \sum_{i=1}^{\infty} i\lambda_i g_f(Y+i) - Yg_f(Y) \right| \approx 0 \qquad \text{for all } f \in \mathcal{F}_0,$$

which in turn leads to

$$d_{\mathcal{F}_0}\big(\mathscr{L}(Y), \mathscr{L}(X)\big) \approx 0\,.$$

Note that if we choose

$$\mathcal{F}_0 := \mathcal{F}_{\mathrm{W}} := \big\{ f : \mathbb{Z}_+ \to \mathbb{R} : |f(i) - f(j)| \le |i-j| \text{ for all } i,j \in \mathbb{Z}_+ \big\}\,,$$

then the metric $d_{\mathcal{F}_0}$ is the Wasserstein distance d_{W} (cf. Appendix A.1.2).

The following lemma shows that if we choose our class of functions $\mathcal{F}_0$ to be $\mathcal{F}_{\mathrm{W}}$, then the Stein equation (3.7) for the compound Poisson distribution $\mathrm{CP}(\theta, n)$ indeed has bounded solutions.

Lemma 3.5 (Arratia et al. (2003, p. 229))**.** *Let $\theta > 0$ and $n \in \mathbb{N}$. Let $X \sim \mathrm{CP}(\theta, n)$. For every function $f \in \mathcal{F}_{\mathrm{W}}$, the Stein equation (3.7), that is in this case*

$$f(k) - \mathbb{E}f(X) = \theta \sum_{i=1}^{n} g(k+i) - kg(k) \qquad \text{for all } k \in \mathbb{Z}_+, \tag{3.8}$$

has a bounded solution $g_f : \mathbb{Z}_+ \to \mathbb{R}$. More precisely, we have

$$|g_f(k)| \le 1 \qquad \text{for all } k \in \mathbb{N} \text{ and } f \in \mathcal{F}_{\mathrm{W}}.$$

3.1.3 Upper bounds for the Wasserstein distance

Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables such that $\mathbb{E}Z_i < \infty$ for all $i \in \mathbb{N}$. As in Subsection 2.1.1 we assume that there is a positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ and a family $\{Z_{ij} : i \in \mathbb{N} \text{ and } 1 \le j \le r_i\}$ of independent $\mathbb{Z}_+$-valued random variables, such that $Z_{i1}, \ldots, Z_{ir_i}$ are identically distributed for each $i \in \mathbb{N}$, and such that $Z_i = \sum_{j=1}^{r_i} Z_{ij}$ for all $i \in \mathbb{N}$. To save on notation we write $\theta_i := i\mathbb{E}Z_i$ for all $i \in \mathbb{N}$ and $\theta_{\sup} := \sup_{i\in\mathbb{N}} i\mathbb{E}Z_i$. Also recall the notation $\varepsilon_{ik}(\theta_i, r_i)$ and $\mu_i(\mathfrak{r})$ from (2.3) and (2.6), respectively.

Let $\{Z_i^*\}_{i\in\mathbb{N}}$ be a sequence of Poisson distributed random variables such that $\mathbb{E}Z_i^* = \mathbb{E}Z_i$ for all $i \in \mathbb{N}$. For every $a \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ we define

$$T_{a,n} := \sum_{i=a+1}^{n} iZ_i \qquad \text{and} \qquad T_{a,n}^* := \sum_{i=a+1}^{n} iZ_i^*\,,$$

and also

$$T_{a,n}^{(i)} := T_{a,n} - iZ_{i1} \qquad \text{for all } a < i \le n.$$

Lemma 3.6. *Let $\theta > 0$. Let $a \in \mathbb{Z}_+$ and $n \in \mathbb{N}$. Then it follows for every bounded function $g : \mathbb{Z}_+ \to \mathbb{R}$ that*

$$\mathbb{E}\big\{T_{a,n}\, g(T_{a,n})\big\} - \theta \sum_{i=a+1}^{n} \mathbb{E}g(T_{a,n}+i) = K_{a,n}^{(1)}(\mathfrak{r}, g) + K_{a,n}^{(2)}(\theta, g)\,, \tag{3.9}$$

where

$$\begin{aligned} K_{a,n}^{(1)}(\mathfrak{r}, g) := &\sum_{i=a+1}^{n} \theta_i \sum_{k=1}^{\infty} k\varepsilon_{ik}(\theta_i, r_i)\, \mathbb{E}g(T_{a,n}^{(i)} + ik) \\ &+ \sum_{i=a+1}^{n} \frac{\theta_i^2}{ir_i} \sum_{k=1}^{\infty} \varepsilon_{ik}(\theta_i, r_i)\big(\mathbb{E}g(T_{a,n}^{(i)} + i) - \mathbb{E}g(T_{a,n}^{(i)} + i(k+1))\big) \\ &+ \sum_{i=a+1}^{n} \frac{\theta_i^2}{ir_i}\big(\mathbb{E}g(T_{a,n}^{(i)} + i) - \mathbb{E}g(T_{a,n}^{(i)} + 2i)\big)\,. \end{aligned} \tag{3.10}$$

and

$$K^{(2)}_{a,n}(\mathfrak{r},g) := \sum_{i=a+1}^{n} (\theta_i - \theta)\mathbb{E}g(T_{a,n}+i)\,. \tag{3.11}$$

Proof. To prove that

$$\mathbb{E}\big\{T_{a,n}\,g(T_{a,n})\big\} - \sum_{i=a+1}^{n} \theta_i \mathbb{E}g(T_{a,n}+i) = K^{(1)}_{a,n}(\mathfrak{r},g)\,, \tag{3.12}$$

the arguments of Arratia et al. (2003, Lemma 9.7) can be followed step by step. Then (3.9) follows immediately from (3.12). □

The following lemma shows that, informally speaking, the term $K^{(1)}_{a,n}(\mathfrak{r},g)$ controls the deviation of the distribution of $T_{a,n}$ from the compound Poisson distribution of the random variable $T^*_{a,n}$, which, by construction has the same expected value as $T_{a,n}$.

Lemma 3.7. *Let $\theta > 0$. Let $a \in \mathbb{Z}_+$ and $n \in \mathbb{N}$. It follows that*

$$\mathscr{L}(T_{a,n}) = \mathscr{L}(T^*_{a,n}) \quad \Leftrightarrow \quad K^{(1)}_{a,n}(\mathfrak{r},g) = 0 \text{ for all bounded } g : \mathbb{Z}_+ \to \mathbb{R}.$$

The inequality

$$|K^{(1)}_{a,n}(\mathfrak{r},g)| \le 5\|g\|(1 \vee \theta_{\sup})^2 \sum_{i=a}^{n-1} \Big(\mu_i(\mathfrak{r}) \vee \frac{1}{(i+1)r_{i+1}}\Big) \tag{3.13}$$

holds for every bounded function $g : \mathbb{Z}_+ \to \mathbb{R}$.

Proof. The first part follows from Lemma 3.2 and equation (3.12). The second part, inequality (3.13), is an implication of the definition of $K^{(1)}_{a,n}(\mathfrak{r},g)$ in (3.10) and the definition of $\mu_i(\mathfrak{r})$ in (2.6). □

In contrast to $K^{(1)}_{a,n}(\mathfrak{r},g)$, the second term $K^{(2)}_{a,n}(\mathfrak{r},g)$ controls the deviation of the expected value of $T_{a,n}$ from $\theta(n-a)$, without having an influence on the "overall" distribution of $T_{a,n}$.

Lemma 3.8. *Let $\theta > 0$. Let $a \in \mathbb{Z}_+$ and $n \in \mathbb{N}$. We have*

$$i\mathbb{E}Z_i = \theta \text{ for all } a < i \le n \quad \Leftrightarrow \quad K^{(2)}_{a,n}(\theta,g) = 0 \text{ for all bounded } g : \mathbb{Z}_+ \to \mathbb{R}.$$

Let $b \in \mathbb{Z}_+$, $b \ge a$. For every bounded function $g : \mathbb{Z}_+ \to \mathbb{R}$ and every $m \in \mathbb{N}$ the following inequality is satisfied:

$$\begin{aligned}|K^{(2)}_{a,n}(\theta,g)| \le 2mn\|g\|\theta'_{\sup} d_{\mathrm{TV}}\big(\mathscr{L}(T_{b,n}), \mathscr{L}(T_{b,n}+1)\big)\\ + 2m\|g\|\theta'_{\sup} + \|g\|\tilde{\theta}_n(m,\theta)n\,,\end{aligned} \tag{3.14}$$

with $\theta'_{\sup} := \theta \vee \theta_{\sup}$ *and*

$$\tilde{\theta}_n(m,\theta) := \max_{0 \le j \le \lfloor n/m \rfloor} \left| \frac{1}{m} \sum_{i=1}^{m} (jm+i)\mathbb{E}Z_{jm+i} - \theta \right|. \tag{3.15}$$

Proof. The first part follows directly from the definition of $K^{(2)}_{a,n}(\theta, g)$ in (3.11). Inequality (3.14) follows from Lemma A.17 (i), setting $X := T_{a,n}$, $x_i := \theta_i$ for all $i \in \mathbb{N}$ and $x := \theta$, and because $a \le b$, from

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{a,n}), \mathscr{L}(T_{a,n}+1)\big) \le d_{\mathrm{TV}}\big(\mathscr{L}(T_{b,n}), \mathscr{L}(T_{b,n}+1)\big). \qquad \square$$

Now we can give an upper bound for the Wasserstein distance between the distribution of $T_{a,n}$ and the compound Poisson distribution $\mathrm{CP}(\theta, n)$.

Proposition 3.9. *Let* $a \in \mathbb{Z}_+$ *and* $n \in \mathbb{N}$. *For every* $m \in \mathbb{N}$ *it follows that*

$$\begin{aligned} d_{\mathrm{W}}\big(\mathscr{L}(T_{a,n}), \mathrm{CP}(\theta,n)\big) \le 5(1 \vee \theta_{\sup})^2 &\sum_{i=0}^{n-1} \Big(\mu_i(\mathfrak{r}) \vee \frac{1}{(i+1)r_{i+1}}\Big) \\ &+ 2mn\theta'_{\sup} d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}), \mathscr{L}(T_{0,n}+1)\big) \\ &+ 2m\theta'_{\sup} + \tilde{\theta}_n(m,\theta)n + a\theta_{\sup}, \end{aligned}$$

with $\theta'_{\sup} := \theta \vee \theta_{\sup}$ *and* $\tilde{\theta}_n(m,\theta)$ *as in* (3.15).

Proof. First, note that

$$d_{\mathrm{W}}\big(\mathscr{L}(T_{a,n}), \mathscr{L}(T_{0,n})\big) \le \mathbb{E}T_{0,a} \le a\theta_{\sup}.$$

From Lemmas 3.5 and 3.6 we conclude that

$$\begin{aligned} d_{\mathrm{W}}\big(\mathscr{L}(T_{0,n}), \mathrm{CP}(\theta,n)\big) &= \sup_{f \in \mathcal{F}_{\mathrm{W}}} \left| \theta \sum_{i=1}^{n} \mathbb{E}g_f(T_{0,n}+i) - \mathbb{E}\big\{T_{0,n} g_f(T_{0,n})\big\} \right| \\ &\le \sup\nolimits_{f \in \mathcal{F}_{\mathrm{W}}} |K^{(1)}_{0,n}(\mathfrak{r}, g_f)| + \sup\nolimits_{f \in \mathcal{F}_{\mathrm{W}}} |K^{(2)}_{0,n}(\theta, g_f)|, \end{aligned}$$

with $\|g_f\| \le 1$ for all $f \in \mathcal{F}_{\mathrm{W}}$. We invoke (3.13) and (3.14) with $a = b = 0$. $\square$

3.2 Global Dickman approximation

3.2.1 The generalized Dickman distribution

We refer to Penrose and Wade (2004) and Arratia et al. (2003, Section 4.2) for a comprehensive overview on the Dickman distribution.

Definition 3.10. Let $\theta > 0$. An $\mathbb{R}_+$-valued random variable X has the *generalized Dickman distribution* with parameter $\theta > 0$ if the Laplace transform of X satisfies

$$\mathbb{E}e^{-tX} = \exp\left(-\int_0^1 \left(1 - e^{-tx}\right)\frac{\theta}{x}\,dx\right) \qquad \text{for all } t > 0. \tag{3.16}$$

We denote this distribution by $\mathrm{GD}(\theta)$.

The $\mathrm{GD}(\theta)$-distribution can be realized a follows. We consider a (scale-invariant) Poisson process on $(0,1)$ with rate θ/x, for $0 < x < 1$, and denote the sequence of arrival times, taken in decreasing order, as $\{\tau_i\}_{i\in\mathbb{N}}$. Then it follows that

$$\mathrm{GD}(\theta) = \mathscr{L}\left(\sum_{i=1}^{\infty} \tau_i\right).$$

Alternatively, if $\{T_i\}_{i\in\mathbb{N}}$ are the successive arrival times of a (translation-invariant) Poisson process on $(0,\infty)$ with constant rate θ, then

$$\mathrm{GD}(\theta) = \mathscr{L}\left(\sum_{i=1}^{\infty} \exp(-T_i)\right).$$

The name of the generalized Dickman distribution comes from its relation to the Dickman function ρ from number theory (Dickman, 1930; Tenenbaum, 1995, Section III, §5.3 and §5.4). This function is defined as the continuous solution of the differential-difference equation

$$x\rho(x) + \rho(x-1) = 0 \qquad \text{for } x > 1,$$

with $\rho(x) = 1$ for all $0 \le x \le 1$. Dickman (1930) showed that

$$\lim_{n\to\infty} \frac{1}{n}\left|\{1 \le i \le n : \text{the largest prime divisor of } i \text{ is} \le n^{1/x}\}\right| = \rho(x).$$

Here, we define a generalized version of this function.

Definition 3.11. Let $\theta > 0$. The *generalized Dickman function* $\rho_\theta : \mathbb{R}_+ \to \mathbb{R}$ is defined as the continuous function which is given by

$$\rho_\theta(x) := \frac{x^{\theta-1}}{\Gamma(\theta)} \qquad \text{for all } 0 \le x \le 1, \tag{3.17}$$

and which satisfies the difference-differential equation

$$x\rho_\theta'(x) + (1-\theta)\rho_\theta(x) + \theta\rho_\theta(x-1) = 0 \qquad \text{for } x > 1. \tag{3.18}$$

If ρ_θ is extended to the whole real axis reals by setting

$$\rho_\theta(x) := 0 \qquad \text{for all } x < 0,$$

then it can be shown that

$$x\rho_\theta(x) = \theta \int_{x-1}^{x} \rho_\theta(t)\,dt \qquad \text{for all } x \in \mathbb{R}, \tag{3.19}$$

and that

$$\int_{-\infty}^{\infty} \rho_\theta(t)\,dt = \mathrm{e}^{-\theta\gamma}\,.$$

The Dickman distribution $\mathrm{GD}(\theta)$ has a density function p_θ, which can be calculated explicitly (Arratia et al., 2003, Lemma 4.7). The density function p_θ equals, up to a multiplicative constant, the generalized Dickman function ρ_θ:

$$p_\theta(x) = \mathrm{e}^{-\theta\gamma}\rho_\theta(x) \qquad \text{for all } x \in \mathbb{R}. \tag{3.20}$$

3.2.2 Wasserstein approximation

The definition of the compound Poisson distribution $\mathrm{CP}(\theta, n)$ and the Dickman distribution $\mathrm{GD}(\theta)$ via the Laplace transformations suggest that the scaled distribution $n^{-1}\mathrm{CP}(\theta, n)$ is close to $\mathrm{GD}(\theta)$ for large n. Indeed, the following lemma can be proved.

Lemma 3.12 (Arratia et al. (2003, Theorem 11.10))**.** *Let $\theta > 0$. Then it follows that*

$$d_{\mathrm{W}}\big(n^{-1}\mathrm{CP}(\theta, n), \mathrm{GD}(\theta)\big) \le \frac{(1+\theta)^2}{n} \qquad \textit{for all } n \in \mathbb{N}.$$

We assume that the conditions and notations from the beginning of Subsection 3.1.3 are in force.

Proposition 3.13. *Let $\theta > 0$. Let $a \in \mathbb{Z}_+$ and $n \in \mathbb{N}$. For every $m \in \mathbb{N}$ it follows that*

$$\begin{aligned} d_{\mathrm{W}}\big(\mathscr{L}(n^{-1}T_{a,n}), \mathrm{GD}(\theta)\big) \le{} & \frac{5}{n}(1 \vee \theta_{\sup})^2 \sum_{i=0}^{n-1}\Big(\mu_i(\mathfrak{r}) \vee \frac{1}{(i+1)r_{i+1}}\Big) \\ & + 2m\theta'_{\sup} d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}), \mathscr{L}(T_{0,n}+1)\big) \\ & + \frac{2m}{n}\theta'_{\sup} + \tilde{\theta}_n(m,\theta) + \frac{a}{n}\theta_{\sup} + \frac{(1+\theta)^2}{n}\,, \end{aligned}$$

with $\tilde{\theta}_n(m,\theta)$ defined as in (3.15)*, and with $\theta'_{\sup} := \theta \vee \theta_{\sup}$.*

Proof. The triangle inequality implies

$$d_{\mathrm{W}}\big(\mathscr{L}(n^{-1}T_{a,n}), \mathrm{GD}(\theta)\big) \le \frac{1}{n}\, d_{\mathrm{W}}\big(\mathscr{L}(T_{a,n}), \mathrm{CP}(\theta, n)\big) \\ + d_{\mathrm{W}}\big(n^{-1}\mathrm{CP}(\theta, n), \mathrm{GD}(\theta)\big)\,.$$

The proposition follows from Proposition 3.9 and Lemma 3.12. □

Theorem 3.14. *Let $\theta > 0$. Let $\mathfrak{r}$ be a sequence of positive integers. Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables that satisfies condition $\mathsf{SUQLC}(\theta, \mathfrak{r})$. Then it follows that*

$$\lim_{n\to\infty} d_{\mathrm{W}}\big(\mathscr{L}(n^{-1}T_{a_n,n}), \mathrm{GD}(\theta)\big) = 0$$

for any non-negative integer sequence $\{a_n\}_{n\in\mathbb{N}}$ that satisfies $\lim_{n\to\infty} a_n/n = 0$.

Proof. We define

$$m_n := \left\lfloor \frac{1}{\sqrt{d_{\mathrm{TV}}(\mathscr{L}(T_{0,n}), \mathscr{L}(T_{0,n}+1))}} \wedge \sqrt{n} \right\rfloor \qquad \text{for all } n \in \mathbb{N}.$$

Since $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition SC it follows that

$$\lim_{n\to\infty} d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}), \mathscr{L}(T_{0,n}+1)\big) = 0\,,$$

and thus that $m_n \to \infty$. Because also $m_n = \mathrm{o}(n)$, and because $\theta_i = i\mathbb{E}Z_i$, $i \in \mathbb{N}$, is $\mathsf{A}(\theta)$-convergent by definition we conclude that $\tilde{\theta}_n(m_n, \theta) \to 0$, with $\tilde{\theta}_n(m_n, \theta)$ defined in (3.15). In particular, $\theta_{\sup} < \infty$. Under condition $\mathsf{UC}(\mathfrak{r})$ we have $\mu_i(\mathfrak{r}) \to 0$. Therefore,

$$\frac{1}{n}\sum_{i=0}^{n-1}\Big(\mu_i(\mathfrak{r}) \vee \frac{1}{(i+1)r_{i+1}}\Big) \xrightarrow{n\to\infty} 0\,.$$

The theorem now follows from Proposition 3.13. □

Theorem 3.15. *Let $\theta > 0$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a non-negative integer sequence. We make the assumption that, for a positive integer sequence $\mathfrak{r}$ and constants $\alpha_1, \alpha_2, \alpha_3 > 0$,*

$$\mu_i(\mathfrak{r}) \vee \frac{1}{ir_i} = \mathrm{O}\Big(\frac{1}{i^{\alpha_1}}\Big)\,, \tag{3.21}$$

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}), \mathscr{L}(T_{0,n}+1)\big) = \mathrm{O}\Big(\frac{1}{n^{\alpha_2}}\Big)\,, \tag{3.22}$$

$$\tilde{\theta}_n(m_n, \theta) = \mathrm{O}\Big(\frac{1}{m_n^{\alpha_3}}\Big) \quad \textit{for all positive } \{m_n\}_{n\in\mathbb{N}} \textit{ with } \lim_{n\to\infty} m_n = \infty. \tag{3.23}$$

It follows that

$$d_{\mathrm{W}}\big(\mathscr{L}(n^{-1}T_{a_n,n}),\mathrm{GD}(\theta)\big) \ll \frac{h_n(1\wedge\alpha_1)}{n} + \Big(\frac{1}{n}\Big)^{\frac{(1\wedge\alpha_2)\alpha_3}{1+\alpha_3}} + \frac{a_n}{n}\,,$$

where $h_n(x) := n^{1-x}$ *if* $0 < x < 1$ *and* $h_n(1) := \log n$.

Proof. Proposition 3.13 implies that for any positive integer sequence $\{m_n\}_{n\in\mathbb{N}}$

$$d_{\mathrm{W}}\big(\mathscr{L}(n^{-1}T_{a_n,n}),\mathrm{GD}(\theta)\big) \ll \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i^{1\wedge\alpha_1}} + \frac{m_n}{n^{1\wedge\alpha_2}} + \frac{1}{m_n^{\alpha_3}} + \frac{a_n}{n}\,.$$

The first summand from the right hand side can be bounded with standard integral estimates; to obtain optimal convergence rates for the second and third summand we choose $m_n \asymp n^{(1\wedge\alpha_2)/(1+\alpha_3)}$. $\square$

Remark 3.16. Condition (3.21) holds for Poisson distributed random variables Z_i for any $\alpha_1 > 0$ without further restrictions, because the r_i can be chosen arbitrarily large in Lemma 2.5. In the binomial case of Lemma 2.6 we have $\mu_i(\mathfrak{r}) = 0$ and thus $\alpha_1 \geq 1$. If the Z_i have negative binomial distributions as in Lemma 2.7, (3.21) holds at least with $\alpha_1 = 1\wedge\varepsilon$, if the probabilities p_i satisfy $p_i \leq 1/i^{\varepsilon}$.

Conditions when (3.22) is satisfied, with some $0 < \alpha_2 < 1$, are given in Section 6. A typical example is given by Theorem 6.4, where the $i\mathbb{E}Z_i$ assumed to be the integer skeleton of a sinusoidal function as in (6.4), for example $i\mathbb{E}Z_i = \theta(1+\cos i)$ as in Example 2.36.

Finally, (3.23) holds true with $\alpha_3 = 1$ again if $i\mathbb{E}Z_i$ is the integer skeleton of a sinusoidal function as in (6.4) (cf. Lemma 2.22). For other examples we refer to Lemma 2.20 and Lemma 2.21.

If we are only interested in distributional approximation of $\mathscr{L}(n^{-1}T_{a_n,n})$ by $\mathrm{GD}(\theta)$, our assumptions in Theorem 3.14 can be significantly relaxed.

Lemma 3.17. *Assume that* $\{Z_i\}_{i\in\mathbb{N}}$ *satisfies the uniformity condition* $\mathsf{UC}(\mathfrak{r})$ *for* $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$*, such that the sequence* $\theta_i = i\mathbb{E}Z_i$, $i\in\mathbb{N}$*, is bounded.*

(i) If $\{\theta_i\}_{i\in\mathbb{N}}$ *converges in the sense of Cesàro to* $\theta > 0$*, that is*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\theta_i = \theta\,, \tag{3.24}$$

then it follows that

$$\lim_{n\to\infty}\mathscr{L}\big(n^{-1}T_{a_n,n}\big) = \mathrm{GD}(\theta) \tag{3.25}$$

for every non-negative integer sequence $a_n = \mathrm{o}(n)$.

(ii) On the other hand, if (3.25) *holds for any non-negative integer sequence* $a_n = \mathrm{o}(n)$*, then* (3.24) *follows.*

Proof. Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent Poisson random variables with expectations $\mathbb{E}\{Z\}_i^* = \theta_i/i$. Set $T_{a,n}^* := \sum_{i=1}^n iZ_i^*$. Using arguments similar to those of the proof of Arratia et al. (2003, Lemma 11.2 (2)), one can show that

$$d_{\mathrm{W}}\Big(\mathscr{L}\big(T_{a,n}\big), \mathscr{L}\big(T_{a,n}^*\big)\Big) \le \theta_{\sup} \sum_{i=a+1}^{n} \Big(2\mu_{i-1}(\mathfrak{r}) + \frac{\theta_{\sup}}{ir_i}\Big),$$

for all $a \in \mathbb{Z}_+$, $n \in \mathbb{N}$. Now the lemma follows from Arratia et al. (1995, Lemma 7). $\square$

3.3 Local Dickman approximation

We adopt the notation from Subsection 3.1.3. That is, let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables such that $\mathbb{E}Z_i < \infty$ for all $i \in \mathbb{N}$, and let $\{Z_i^*\}_{i\in\mathbb{N}}$ be a sequence of Poisson distributed random variables such that $\mathbb{E}Z_i^* = \mathbb{E}Z_i$ for all $i \in \mathbb{N}$. We also assume that there is a positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ and a family $\{Z_{ij} : i \in \mathbb{N} \text{ and } 1 \le j \le r_i\}$ of independent $\mathbb{Z}_+$-valued random variables, such that $Z_{i1}, \dots, Z_{ir_i}$ are identically distributed for each $i \in \mathbb{N}$, and such that $Z_i = \sum_{j=1}^{r_i} Z_{ij}$ for all $i \in \mathbb{N}$. Also, set $\theta_i := i\mathbb{E}Z_i$ for all $i \in \mathbb{N}$, and $\theta_{\sup} := \sup_{i\in\mathbb{N}} \theta_i$.

Lemma 3.18. *Let*

$$l_0 := \min\big\{i \in \mathbb{N} : \mathbb{P}[Z_{j1} = 0] \ge 1/2 \textit{ for all } j \ge i\big\}, \tag{3.26}$$

and

$$C(l_0) := 2 \vee \max_{1\le i\le l_0} \left(\frac{1}{\sup_{j\in\mathbb{Z}_+} \mathbb{P}[Z_{i1} = j]}\right). \tag{3.27}$$

Let $a, b \in \mathbb{Z}_+$, $a \le b$, *and* $l, m, n \in \mathbb{N}$, $l > l_0$. *Then it follows for every* $k \in \mathbb{Z}_+$ *that*

$$\begin{aligned}
&\big|k\mathbb{P}[T_{a,n} = k] - \theta\mathbb{P}[k - n \le T_{a,n} < k - a]\big| \\
&\quad \le 2m^2\theta'_{\sup} d_{\mathrm{TV}}\big(\mathscr{L}(T_{b,n}), \mathscr{L}(T_{b,n} + 1)\big) + 2m\theta'_{\sup} \sup\nolimits_{i\in\mathbb{Z}_+} \mathbb{P}\big[T_{a,n} = i\big] \\
&\qquad + \tilde{\theta}_n(m, \theta) + 5(1 \vee \theta_{\sup})^2 \bigg(C(l_0) \sup\nolimits_{j\in\mathbb{Z}_+} \mathbb{P}\big[T_{a,n} = j\big] \sum_{i=a}^{l-1} \Big(\mu_i(\mathfrak{r}) \vee \frac{1}{i+1}\Big) \\
&\qquad + 10\Big(\mu_l(\mathfrak{r}) \vee \frac{1}{l+1}\Big)\bigg),
\end{aligned}$$

with $\theta'_{\sup} := \theta \vee \theta_{\sup}$.

Proof. Fix $k \in \mathbb{Z}_+$ and define the function $g_k(j) := \mathbf{1}\{j = k\}$ for $j \in \mathbb{Z}_+$. We invoke Lemma 3.6 with $g := g_k$; equation (3.9) translates into

$$k\mathbb{P}[T_{a,n} = k] - \theta\mathbb{P}[k - n \le T_{a,n} < k - a] = K^{(1)}_{a,n}(\mathfrak{r}, g_k) + K^{(2)}_{a,n}(\theta, g_k)\,. \tag{3.28}$$

First, we give a bound for $K^{(2)}_{a,n}(\theta, g_k)$. Its definition in (3.11), together with Lemma A.17 (ii) and

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{a,n}), \mathscr{L}(T_{a,n} + 1)\big) \le d_{\mathrm{TV}}\big(\mathscr{L}(T_{b,n}), \mathscr{L}(T_{b,n} + 1)\big)\,,$$

leads to

$$\begin{aligned}\big|K^{(2)}_{a,n}(\theta, g_k)\big| &= \Big|\sum_{i=a+1}^{n} (\theta_i - \theta)\mathbb{P}[T_{a,n} + i = k]\Big| \\ &\le 2m^2\theta'_{\sup} d_{\mathrm{TV}}\big(\mathscr{L}(T_{b,n}), \mathscr{L}(T_{b,n} + 1)\big) \\ &\quad + 2m\theta'_{\sup} \sup\nolimits_{j\in\mathbb{Z}_+} \mathbb{P}[T_{a,n} = j] + \tilde{\theta}_n(m, \theta)\,.\end{aligned}$$

To bound $K^{(1)}_{a,n}(\mathfrak{r}, g_k)$ we note that due to the special form of g_k we obtain a first bound directly from its definition in (3.10), namely

$$\begin{aligned}\big|K^{(1)}_{a,n}(\mathfrak{r}, g_k)\big| \le \theta_{\sup} &\underbrace{\sum_{i=a+1}^{n}\sum_{j=1}^{\infty} j\big|\varepsilon_{ij}(\theta_i, r_i)\big|\mathbb{P}\big[T^{(i)}_{a,n} = k - ij\big]}_{U_1} \\ &+ \theta_{\sup}^2 \underbrace{\sum_{i=a+1}^{n} \frac{1}{i} \sum_{j=1}^{\infty}\big|\varepsilon_{ij}(\theta_i, r_i)\big|\mathbb{P}[T^{(i)}_{a,n} = k - i]}_{U_2} \\ &+ \theta_{\sup}^2 \underbrace{\sum_{i=a+1}^{n} \frac{1}{i} \sum_{j=1}^{\infty}\big|\varepsilon_{ij}(\theta_i, r_i)\big|\mathbb{P}[T^{(i)}_{a,n} = k - ij - i]}_{U_3} \\ &+ \theta_{\sup}^2 \underbrace{\sum_{i=a+1}^{n} \frac{1}{i}\big(\mathbb{P}[T^{(i)}_{a,n} = k - i] + \mathbb{P}[T^{(i)}_{a,n} = k - 2i]\big)}_{U_4}\,.\end{aligned}$$

We have

$$\mathbb{P}[T^{(i)}_{a,n} = k]\mathbb{P}[Z_{i1} = j] \le \mathbb{P}[T_{a,n} = k + ij] \qquad \text{for all } i \in \mathbb{N},\ j, k \in \mathbb{Z}_+.$$

If $i > l_0$ and $j = 0$ this yields

$$\mathbb{P}[T_{a,n}^{(i)} = k] \le 2\mathbb{P}[T_{a,n} = k] \qquad \text{for all } k \in \mathbb{Z}_+. \tag{3.29}$$

If $1 \le i \le l_0$, we conclude that

$$\begin{aligned}\mathbb{P}[T_{a,n}^{(i)} = k] &\le \frac{\sup_{j\in\mathbb{Z}_+} \mathbb{P}[T_{a,n} = k + ij]}{\sup_{j\in\mathbb{Z}_+} \mathbb{P}[Z_{i1} = j]} \\ &\le \max_{1\le i\le l_0} \left(\frac{1}{\sup_{j\in\mathbb{Z}_+} \mathbb{P}[Z_{i1} = j]}\right) \sup_{j\in\mathbb{Z}_+} \mathbb{P}[T_{a,n} = j]\,.\end{aligned} \tag{3.30}$$

The inequalities (3.29) and (3.30) imply

$$\sup_{k\in\mathbb{Z}_+} \mathbb{P}\big[T_{a,n}^{(i)} = k\big] \le C(l_0) \sup_{k\in\mathbb{Z}_+} \mathbb{P}\big[T_{a,n} = k\big] \qquad \text{for all } i \in \mathbb{N}\,. \tag{3.31}$$

The inequalities (3.29) and (3.31) can be used to bound each summand U_1, U_2, U_3 and U_4 separately. This is accomplished by the same arguments as can be found in the proof of Arratia et al. (2003, Theorem 12.1). As illustration, we give the bound for U_1, which we split into two parts, one where $1 \le i \le l$ and one where $l < i < n$, for any $l > l_0$. Then we apply (3.31) to the first, and (3.29) to the second part:

$$\begin{aligned}&\sum_{i=a+1}^{n} \sum_{j=1}^{\infty} j\big|\varepsilon_{ij}(\theta_i, r_i)\big|\mathbb{P}\big[T_{a,n}^{(i)} = k - ij\big] \\ &\quad = \sum_{i=a+1}^{l} \sum_{j=1}^{\infty} j\big|\varepsilon_{ij}(\theta_i, r_i)\big|\mathbb{P}\big[T_{a,n}^{(i)} = k - ij\big] \\ &\qquad + \sum_{i=l+1}^{n} \sum_{j=1}^{\infty} j\big|\varepsilon_{ij}(\theta_i, r_i)\big|\mathbb{P}\big[T_{a,n}^{(i)} = k - ij\big] \\ &\quad \le C(l_0) \sup_{j\in\mathbb{Z}_+} \mathbb{P}\big[T_{a,n} = j\big] \sum_{i=a+1}^{l} \sum_{j=1}^{\infty} j\big|\varepsilon_{ij}(\theta_i, r_i)\big| \\ &\qquad + 2\sum_{j=1}^{\infty} j \sup_{i>l}\big|\varepsilon_{ij}(\theta_i, r_i)\big| \sum_{i=l+1}^{n} \mathbb{P}\big[T_{a,n} = k - ij\big] \\ &\quad \le C(l_0) \sup_{j\in\mathbb{Z}_+} \mathbb{P}\big[T_{a,n} = j\big] \sum_{i=a}^{l-1} \mu_i(\mathfrak{r}) + 2\mu_l(\mathfrak{r})\,.\end{aligned}$$

The estimation of U_2 and U_3 is of the same kind. Here, we obtain the bound

$$U_r \le C(l_0) \sup_{j\in\mathbb{Z}_+} \mathbb{P}\big[T_{a,n} = j\big] \sum_{i=a}^{l-1} \frac{\mu_i(\mathfrak{r})}{i+1} + \frac{2\mu_l(\mathfrak{r})}{l+1} \qquad \text{for } r = 2, 3.$$

For the final term we obtain

$$U_4 \le 2C(l_0)\sup_{j\in\mathbb{Z}_+}\mathbb{P}\big[T_{a,n}=j\big]\sum_{i=a}^{l-1}\frac{1}{i+1}+\frac{4}{l+1}\,.$$

Collecting the bounds of U_1, U_2, U_3 and U_4 yields

$$\begin{aligned}\big|K^{(1)}_{a,n}(\mathfrak{r},g_k)\big| \le 5(1\vee\theta_{\sup})^2\bigg(&C(l_0)\sup_{j\in\mathbb{Z}_+}\mathbb{P}\big[T_{a,n}=j\big]\sum_{i=a}^{l-1}\Big(\mu_i(\mathfrak{r})\vee\frac{1}{i+1}\Big)\\ &+10\Big(\mu_l(\mathfrak{r})\vee\frac{1}{l+1}\Big)\bigg)\,.\end{aligned} \tag{3.32}$$

This proves the lemma. □

Lemma 3.19. *Let $a\in\mathbb{Z}_+$ and $n\in\mathbb{N}$.*

(i) If the Poisson random variables $\{Z_i^*\}_{i\in\mathbb{N}}$ are mutually independent, then we have

$$\sup_{k\in\mathbb{Z}_+}\mathbb{P}\big[T^*_{a,n}=k\big]\le\exp\Big(-(1\vee\theta_{\sup})^{-1}\sum_{i=a+1}^{n}\mathbb{E}Z_i\Big)\,. \tag{3.33}$$

(ii) For our sequence $\{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables, we have, for any $b\in\mathbb{Z}_+$ with $a\le b$,

$$\begin{aligned}\sup_{k\in\mathbb{Z}_+}\mathbb{P}\big[T_{a,n}=k\big]\le\sup_{k\in\mathbb{Z}_+}\mathbb{P}\big[T^*_{b,n}=k\big]+3\sup_{i\ge b}\big|\varepsilon_{i1}(\theta_i,r_i)\big|\sum_{i=b+1}^{n}\mathbb{E}Z_i\\ +\sum_{i=b+1}^{n}\big((\mathbb{E}Z_i)^2+2(\mathbb{E}Z_i-1)^+\big)\,.\end{aligned} \tag{3.34}$$

If $\{Z_i\}_{i\in\mathbb{N}}$ satisfies the quasi-logarithmic condition $\mathsf{SUQLC}(\theta,\mathfrak{r})$ *we have*

$$\lim_{n\to\infty}\sup_{k\in\mathbb{Z}_+}\mathbb{P}\big[T_{a_n,n}=k\big]=0 \tag{3.35}$$

for every non-negative integer sequence $\{a_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}a_n/n=0$. If in addition $\varepsilon_{i1}(\theta_i,r_i)=\mathrm{O}(1/i^\alpha)$ for some $\alpha>0$, it follows that

$$\sup_{k\in\mathbb{Z}_+}\mathbb{P}\big[T_{a_n,n}=k\big]=\mathrm{O}\bigg(\Big(\frac{a_n\vee 1}{n}\Big)^{\frac{\alpha\wedge(\theta/(\theta\vee 1))}{4}}\bigg)\,. \tag{3.36}$$

Proof. The arguments of the proof are similar to those of Arratia et al. (2003, Lemma 6.10 and Lemma 4.12).

(i) Corollary 3.3 yields

$$k\mathbb{P}\big[T^*_{a,n} = k\big] = \sum_{i=a+1}^{n} i\mathbb{E}Z_i\,\mathbb{P}\big[T^*_{a,n} = k-i\big] \qquad \text{for all } k \in \mathbb{N}.$$

Assume that $\theta_{\sup} = \sup_{j\in\mathbb{N}} j\mathbb{E}Z_j \leq 1$. Then we can conclude that

$$\mathbb{P}\big[T^*_{a,n} = k\big] \leq \frac{1}{k}\sum_{j=0}^{k-1} \mathbb{P}\big[T^*_{a,n} = j\big] \qquad \text{for all } k \in \mathbb{N}.$$

We obtain further, invoking also the independence of $\{Z^*_i\}_{i\in\mathbb{N}}$, that

$$\sup_{k\in\mathbb{Z}_+} \mathbb{P}\big[T^*_{a,n} = k\big] \leq \mathbb{P}\big[T^*_{a,n} = 0\big] = \prod_{i=a+1}^{n} \mathbb{P}[Z^*_i = 0] \leq \exp\Bigg(-\sum_{i=a+1}^{n} \mathbb{E}Z_i\Bigg).$$

Now assume that $1 < \theta_{\sup} < \infty$. Consider mutually independent Poisson random variables $\{Z'_i\}_{i\in\mathbb{N}}$ and $\{Z''_i\}_{i\in\mathbb{N}}$ with

$$\mathbb{E}Z'_i = \frac{\mathbb{E}Z_i}{\theta_{\sup}} \quad \text{and} \quad \mathbb{E}Z''_i = \frac{\mathbb{E}Z_i(\theta_{\sup}-1)}{\theta_{\sup}} \qquad \text{for all } i \in \mathbb{N}.$$

We set

$$T'_{a,n} := \sum_{i=a+1}^{n} iZ'_i \quad \text{and} \quad T''_{a,n} := \sum_{i=a+1}^{n} iZ'_i\,.$$

Then we can write

$$T^*_{a,n} = T'_{a,n} + T''_{a,n}\,,$$

so that we obtain, because of $\sup_{j\in\mathbb{N}} j\mathbb{E}Z'_j \leq 1$, that

$$\begin{aligned}
\mathbb{P}\big[T^*_{a,n} = k\big] &= \sum_{i=0}^{k} \mathbb{P}\big[T'_{a,n} = i\big]\mathbb{P}\big[T''_{a,n} = k-i\big] \\
&\leq \sup_{j\in\mathbb{Z}_+} \mathbb{P}\big[T'_{a,n} = j\big] \\
&\leq \exp\Bigg(-\sum_{i=a+1}^{n} \mathbb{E}Z'_i\Bigg) \\
&\leq \exp\Bigg(-\frac{1}{\theta_{\sup}}\sum_{i=a+1}^{n} \mathbb{E}Z_i\Bigg) \qquad \text{for all } k \in \mathbb{Z}_+.
\end{aligned}$$

Finally, if $\theta_{\sup} = \infty$ the right hand side of (3.33) is equal to 1.

(ii) We turn to the general case, considering the random variable $T_{a,n}$. Because $a \le b$, we can write $T_{a,n} = T_{a,b} + T_{b,n}$, and we obtain with a convolution argument as above

$$\sup_{k\in\mathbb{Z}_+} \mathbb{P}\big[T_{a,n} = k\big] \le \sup_{k\in\mathbb{Z}_+} \mathbb{P}\big[T_{b,n} = k\big]\,.$$

We compare point probabilities of $T_{b,n}$ and $T^*_{b,n}$:

$$\big|\mathbb{P}[T_{b,n} = k] - \mathbb{P}[T^*_{b,n} = k]\big| \le \sum_{i=b+1}^{n} d_{\mathrm{TV}}\big(\mathscr{L}(Z_i), \mathscr{L}(Z^*_i)\big) \qquad \text{for all } k \in \mathbb{Z}_+.$$

Then we invoke the triangle inequality together with (A.13) and conclude that

$$\begin{aligned} d_{\mathrm{TV}}\big(\mathscr{L}(Z_i), \mathscr{L}(Z^*_i)\big) &\le d_{\mathrm{TV}}\big(\mathscr{L}(Z_i), \mathrm{Be}(1\wedge\mathbb{E}Z_i)\big) + d_{\mathrm{TV}}\big(\mathrm{Be}(1\wedge\mathbb{E}Z_i), \mathscr{L}(Z^*_i)\big) \\ &\le 3\mathbb{E}Z_i\big(\big|\varepsilon_{i1}(\theta_i, r_i)\big| + \mathbb{E}Z_i\big) + 2(\mathbb{E}Z_i - 1\wedge\mathbb{E}Z_i)\,. \end{aligned}$$

for $a < i \le n$. Hence,

$$\begin{aligned} \sup_{k\in\mathbb{Z}_+} \mathbb{P}\big[T_{b,n} = k\big] \le \sup_{k\in\mathbb{Z}_+} \mathbb{P}\big[T^*_{b,n} = k\big] + 3\sup_{i\ge b}\big|\varepsilon_{i1}(\theta_i, r_i)\big| \sum_{i=b+1}^{n} \mathbb{E}Z_i \\ + \sum_{i=b+1}^{n} \big((\mathbb{E}Z_i)^2 + 2(\mathbb{E}Z_i - 1\wedge\mathbb{E}Z_i)\big)\,. \end{aligned}$$

This implies the first assertion of the second part.

Under condition $\mathsf{SUQLC}(\theta, \mathfrak{r})$ we have $\theta_{\sup} < \infty$. We choose a positive integer sequence $b_n = \mathrm{o}(n)$ such that $b_n \to \infty$ and $a_n \le b_n$ for all $n \in \mathbb{N}$, large enough for

$$\sup_{i\ge b_n}\big|\varepsilon_{i1}(\theta_i, r_i)\big| \sum_{i=b_n+1}^{n} \frac{\theta_i}{i} \ll \sup_{i\ge b_n}\big|\varepsilon_{i1}(\theta_i, r_i)\big| \log(n/b_n) \xrightarrow{n\to\infty} 0\,.$$

Recalling that $\theta_i = i\mathbb{E}Z_i$ for all $i \in \mathbb{N}$, we then also have

$$\sum_{i=b_n+1}^{n} \big((\mathbb{E}Z_i)^2 + 2(\mathbb{E}Z_i - 1)^+\big) \ll \frac{1}{b_n} \xrightarrow{n\to\infty} 0\,,$$

and, invoking Lemma A.16 (iii) and setting $\bar\theta_{\sup} := \theta_{\sup} \vee 1$,

$$\begin{aligned} \exp\Big(-\frac{1}{\bar\theta_{\sup}} \sum_{i=b_n+1}^{n} \frac{\theta_i}{i}\Big) &= \exp\Big(-\frac{1}{\bar\theta_{\sup}} \sum_{i=b_n+1}^{n} \frac{\theta}{i}\Big) \exp\Big(-\frac{1}{\bar\theta_{\sup}} \sum_{i=b_n+1}^{n} \frac{\theta_i - \theta}{i}\Big) \\ &\ll \Big(\frac{b_n}{n}\Big)^{\theta/\bar\theta_{\sup} - \tilde\theta_n(m_n,\theta)/\bar\theta_{\sup}}\,, \end{aligned} \tag{3.37}$$

with a positive integer sequence $m_n = \mathrm{o}(n)$ that satisfies $m_n \to \infty$ and $m_n \le b_n$ for all $n \in \mathbb{N}$. Since $\tilde{\theta}_n(m_n, \theta) \to 0$ under condition $\mathsf{SUQLC}(\theta, \mathfrak{r})$, the right hand side of (3.37) converges to 0, which proves (3.35).

If $\varepsilon_{i1}(\theta_i, r_i) = \mathrm{O}(1/i^{\alpha})$, we can choose $b_n \asymp (n/\bar{a}_n)^{1/2} \vee \bar{a}_n$; (3.36) follows if we examine the cases $(n/\bar{a}_n)^{1/2} \ge \bar{a}_n$ and $(n/\bar{a}_n)^{1/2} < \bar{a}_n$ separately. □

Corollary 3.20. *Let $\theta > 0$. Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables such that*

$$\operatorname*{alim}_{i\to\infty} \theta_i = \theta \qquad \text{and} \qquad \lim_{i\to\infty} \varepsilon_{i1}(\theta_i, r_i) = 0\,,$$

recalling that $\theta_i = i\mathbb{E}Z_i$ for all $i \in \mathbb{N}$. Then it follows that

$$\lim_{n\to\infty} \sup\nolimits_{k\in\mathbb{Z}_+} \mathbb{P}\big[T_{a_n,n} = k\big] = 0$$

for every non-negative integer sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_n/n = 0$.

Proof. By assumption we have $\varepsilon_{i1}(\theta_i, r_i) \to 0$. We therefore may choose sequence of non-negative integers $\{b_n\}_{n\in\mathbb{N}}$ such that $b_n \to \infty$ and $b_n = \mathrm{o}(n)$, and such that

$$\lim_{n\to\infty} \sup\nolimits_{i\ge b_n} \big|\varepsilon_{i1}(\theta_i, r_i)\big| \log(n/b_n) = 0\,. \tag{3.38}$$

Lemma 2.18 implies that $\theta_{\sup} < \infty$. Set $C := (1 \vee \theta_{\sup})^{-1}$. Our assumption $\operatorname{alim}_{i\to\infty} \theta_i = \theta$ entails $\operatorname{alim}_{i\to\infty} C\theta_i = C\theta$ (cf. Definition 2.11 and Lemma 2.12). Then Lemma 2.19 implies that the function

$$\ell(n) := \exp\left(C \sum_{i=1}^{n} \frac{\theta_i - \theta}{i}\right), \qquad n \in \mathbb{N}\,,$$

is slowly varying at infinity. Thus, we find a sequence of positive integers $\{c_n\}_{n\in\mathbb{N}}$ such that $c_n \ge a_n \vee b_n$ for all n large enough (and thus $c_n \to \infty$), $c_n = \mathrm{o}(n)$ and

$$\lim_{n\to\infty} \frac{\ell(c_n)}{\ell(n)} = 1\,.$$

Lemma 3.19 (i) implies that

$$\begin{aligned} \sup\nolimits_{k\in\mathbb{Z}_+} \mathbb{P}\big[T^*_{c_n,n} = k\big] &\le \exp\left(-C \sum_{i=c_n+1}^{n} \frac{\theta}{i}\right) \exp\left(-C \sum_{i=c_n+1}^{n} \frac{\theta_i - \theta}{i}\right) \\ &\le \left(\frac{c_n + 1}{n+1}\right)^{C\theta} \frac{\ell(c_n)}{\ell(n)} \qquad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Because $c_n = \mathrm{o}(n)$ and $\ell(c_n)/\ell(n) \to 1$ by construction, we obtain

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}_+} \mathbb{P}\big[T^*_{c_n,n} = k\big] = 0\,. \tag{3.39}$$

The $\mathsf{A}(\theta)$-convergence of $\{\theta_i\}_{i\in\mathbb{N}}$ yields $\mathbb{E}Z_i = \mathrm{O}(1/i)$. Therefore, for some constant $C' > 0$,

$$\sum_{i=c_n+1}^{n} \mathbb{E}Z_i \le C' \log(n/c_n) \qquad \text{for all } n \in \mathbb{N}.$$

Now (3.38) implies, since $c_n \ge b_n$ for all $n \in \mathbb{N}$, that

$$\lim_{n\to\infty} \sup_{i\ge c_n} \big|\varepsilon_{i1}(\theta_i, r_i)\big| \sum_{i=c_n+1}^{n} \mathbb{E}Z_i = 0\,. \tag{3.40}$$

It is also immediate from $\mathbb{E}Z_i = \mathrm{O}(1/i)$ and from $c_n \to \infty$ that

$$\lim_{n\to\infty} \sum_{i=c_n+1}^{n} \big((\mathbb{E}Z_i)^2 + 2(\mathbb{E}Z_i - 1 \wedge \mathbb{E}Z_i)\big) = 0\,. \tag{3.41}$$

The corollary now follows from Lemma 3.19 (ii) along with (3.39), (3.40) and (3.41). $\square$

Theorem 3.21. *Let $\theta > 0$. Let $\mathfrak{r}$ be a sequence of positive integers and $\mathfrak{b} := \{b_n\}_{n\in\mathbb{N}}$ a sequence of non-negative integers such that $\lim_{n\to\infty} b_n/n = 0$. Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables that satisfies condition $\mathsf{SUQLC}(\theta, \mathfrak{r}, \mathfrak{b})$. Also, let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative integers such that, for some $x > 0$,*

$$\lim_{n\to\infty} \frac{k_n}{n} = x\,.$$

Then it follows that

$$\lim_{n\to\infty} n\mathbb{P}\big[T_{a_n,n} = k_n\big] = p_\theta(x)$$

for every non-negative integer sequence $\{a_n\}_{n\in\mathbb{N}}$ that satisfies $a_n \le b_n$ for all $n \in \mathbb{N}$.

Proof. The triangle inequality immediately yields for every $n \in \mathbb{N}$

$$\begin{aligned}
&\big|n\mathbb{P}\big[T_{a_n,n} = k_n\big] - p_\theta(x)\big| \le \\
&\quad \frac{n}{k_n}\big|k_n\mathbb{P}[T_{a_n,n} = k_n] - \theta\mathbb{P}[k_n - n \le T_{a_n,n} < k_n - a_n]\big| \\
&\quad + \Big|\frac{n}{k_n}\theta\mathbb{P}\big[n^{-1}k_n - 1 \le n^{-1}T_{a_n,n} < n^{-1}k_n - n^{-1}a_n\big] - p_\theta(x)\Big|\,.
\end{aligned}$$

We show that both summands on the right hand side of this inequality converge to 0 as $n \to \infty$.

For the first, we apply Lemma 3.18. For this, first recall that condition $\mathsf{SUQLC}(\theta, \mathfrak{r}, \mathfrak{b})$, by definition, is equivalent to

$$\operatorname*{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta \quad \text{and} \quad \lim_{i\to\infty} \mu_i(\mathfrak{r}) = 0 \tag{3.42}$$

and

$$\lim_{n\to\infty} d_{\mathrm{TV}}\big(\mathscr{L}(T_{b_n,n}), \mathscr{L}(T_{b_n,n}+1)\big) = 0\,. \tag{3.43}$$

The assumptions of Corollary 3.20 are satisfied under (3.42) (cf. the definition of $\mu_i(\mathfrak{r})$ in (2.6)). Therefore,

$$\lim_{n\to\infty} \sup\nolimits_{k\in\mathbb{Z}_+} \mathbb{P}\big[T_{a_n,n} = k\big] = 0\,.$$

We define for all $n \in \mathbb{N}$

$$m_n := \left\lfloor \frac{1}{(d_{\mathrm{TV}}(\mathscr{L}(T_{b_n,n}), \mathscr{L}(T_{b_n,n}+1)))^{1/4}} \wedge \frac{1}{(\sup_{k\in\mathbb{Z}_+} \mathbb{P}\big[T_{a_n,n} = k\big])^{1/2}} \wedge \sqrt{n} \right\rfloor .$$

We have $m_n = \mathrm{o}(n)$ and, recalling (3.43), $m_n \to \infty$. Then the $\mathsf{A}(\theta)$-convergence of $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ implies that $\tilde{\theta}_n(m_n, \theta) \to 0$, with $\tilde{\theta}_n(m_n, \theta)$ defined in (3.15).

Now choose a positive integer sequence $\{l_n\}_{n\in\mathbb{N}}$ such that $l_n > l_0$, with l_0 defined in (3.26), and such that $l_n \to \infty$ slowly enough for

$$\lim_{n\to\infty} \sup\nolimits_{j\in\mathbb{Z}_+} \mathbb{P}\big[T_{a_n,n} = j\big] \sum_{i=1}^{l_n} \Big(\mu_i(\mathfrak{r}) \vee \frac{1}{i+1}\Big) = 0$$

to hold.

Finally, the $\mathsf{A}(\theta)$-convergence also yields $\theta_{\sup} = \sup_{i\in\mathbb{N}} i\mathbb{E}Z_i < \infty$, and we conclude from Lemma 3.18, choosing $a := a_n$, $b := b_n$, $k := k_n$, $l := l_n$ and $m := m_n$, that

$$\big|k_n\mathbb{P}[T_{a_n,n} = k_n] - \theta\mathbb{P}[k_n - n \le T_{a_n,n} < k_n - a_n]\big| = 0\,.$$

Recall that the generalized Dickman distribution $\mathrm{GD}(\theta)$ is absolutely continuous with density p_θ. Theorem 3.14 and (3.19) then imply

$$\lim_{n\to\infty} \frac{n}{k_n} \theta\mathbb{P}\big[n^{-1}k_n - 1 \le n^{-1}T_{a_n,n} < n^{-1}k_n - n^{-1}a_n\big] = \int_{x-1}^{x} p_\theta(t)\,dt = p_\theta(x)\,.$$

This proves the theorem. $\square$

4 Quasi-logarithmic structures

4.1 Random decomposable combinatorial structures

4.1.1 Definition of an RDCS

Definition 4.1. A *random decomposable combinatorial structure (RDCS) of size* n is a triple $\mathcal{C}^{(n)} := (\Omega_n, \nu_n, C^{(n)})$, where $(\Omega_n, \mathcal{P}(\Omega_n), \nu_n)$ is a discrete probability space such that

$$\nu_n\bigl(\{\omega\}\bigr) > 0 \qquad \text{for all } \omega \in \Omega_n, \tag{4.1}$$

and where

$$C^{(n)} := \bigl(C_1^{(n)}, \dots, C_n^{(n)}\bigr) : \bigl(\Omega_n, \mathcal{P}(\Omega_n), \nu_n\bigr) \longrightarrow \bigl(\mathbb{Z}_+^n, \mathcal{P}(\mathbb{Z}_+^n)\bigr)$$

is a random vector that satisfies the *total size conservation law*

$$\mathbb{P}_{\nu_n}\Bigl[\sum\nolimits_{i=1}^{n} iC_i^{(n)} = n\Bigr] = 1\,. \tag{4.2}$$

The random vector $C^{(n)}$ is called *component spectrum* of $\mathcal{C}^{(n)}$. We refer to an element $\omega \in \Omega_n$, or to the whole triple $(\omega, \nu_n(\{\omega\}), C^{(n)}(\omega))$, as an *instance* of an RDCS of size n. The total size conservation law (4.2) allows us to interpret $C_i^{(n)}(\omega)$ as the *number of components of size* i into which $\omega \in \Omega_n$ can be decomposed.

Definition 4.2. A *random decomposable combinatorial structure* is a sequence $\mathfrak{C} := \{\mathcal{C}^{(n)}\}_{n\in\mathbb{N}}$, where $\mathcal{C}^{(n)} := (\Omega_n, \nu_n, C^{(n)})$ is either an RDCS of size n, or where $\Omega_n = \emptyset$, $\nu_n = 0$ and $C^{(n)} = (0, \dots, 0)$.

Note that $\mathcal{C}^{(n)} = (\emptyset, 0, (0, \dots, 0))$ in Definition 4.2 is introduced to allow "empty" RDCS of some sizes; it appears only to give every RDCS $\mathfrak{C}$ the form of a sequence that is indexed by $\mathbb{N}$. The examples of RDCS we encounter have $\Omega_n \neq \emptyset$ for all $n \in \mathbb{N}$, or at least for all n large enough.

4.1.2 Tilting of RDCS

Definition 4.3. Let $H := \{\eta_i\}_{i\in\mathbb{N}}$ be a sequence of non-negative real numbers. We say that H is *admissible to tilt* the RDCS $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$, if

$$0 < \mathbb{E}_{\nu_n}\left\{\prod_{i=1}^{n} \eta_i^{C_i^{(n)}}\right\} < \infty \qquad \text{for all } n \text{ with } \Omega_n \neq \emptyset. \tag{4.3}$$

Here we use that convention that, if $\eta_i = 0$, for any $\omega \in \Omega_n$

$$\eta_i^{C_i^{(n)}(\omega)} := \begin{cases} 0 & \text{if } C_i^{(n)}(\omega) \geq 1, \\ 1 & \text{if } C_i^{(n)}(\omega) = 0. \end{cases} \tag{4.4}$$

Now let $H := \{\eta_i\}_{i\in\mathbb{N}}$ be an arbitrary non-negative sequence that is admissible to tilt $\mathfrak{C}$. The RDCS $\widetilde{\mathfrak{C}} := \{(\widetilde{\Omega}_n, \widetilde{\nu}_n, \widetilde{C}^{(n)})\}_{n\in\mathbb{N}}$ that arises from $\mathfrak{C} = \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ through H*-tilting* is constructed as follows. For any n with $\Omega_n \neq \emptyset$ we define a new probability measure on $\mathcal{P}(\Omega_n)$ by

$$\nu_n^*(\{\omega\}) := \frac{\prod_{i=1}^{n} \eta_i^{C_i^{(n)}(\omega)}}{\mathbb{E}_{\nu_n}\left\{\prod_{i=1}^{n} \eta_i^{C_i^{(n)}}\right\}}\, \nu_n(\{\omega\}) \qquad \text{for all } \omega \in \Omega_n. \tag{4.5}$$

Set

$$\widetilde{\Omega}_n := \left\{\omega \in \Omega_n : \nu_n^*(\{\omega\}) > 0\right\}.$$

If $\widetilde{\Omega}_n \neq \emptyset$, the probability measure $\widetilde{\nu}_n$ is the defined as the restriction of ν_n^* to $\mathcal{P}(\widetilde{\Omega}_n)$, and $\widetilde{C}^{(n)}$ as the restriction of $C^{(n)}$ to $\widetilde{\Omega}_n$. The total size conservation law (4.2) carries over to

$$\mathbb{P}_{\widetilde{\nu}_n}\left[\sum\nolimits_{i=1}^{n} i\widetilde{C}_i^{(n)} = n\right] = 1 \qquad \text{for all } n \text{ with } \widetilde{\Omega}_n \neq \emptyset. \tag{4.6}$$

If $\Omega_n = \emptyset$, we simply set $(\widetilde{\Omega}_n, \widetilde{\nu}_n, \widetilde{C}^{(n)}) := (\emptyset, 0, (0, \dots, 0))$.

For an interpretation of the H-tilting of $\mathfrak{C}$, we assume that, for some $n \in \mathbb{N}$, Ω_n is a non-empty finite set and that ν_n is the uniform measure. Thus, each instance $\omega \in \Omega_n$ is picked with the same probability independent of how its component spectrum $C^{(n)}(\omega)$ looks. If $\mathfrak{C}$ is tilted with an admissible sequence H then in the H-tilted structure $\widetilde{\mathfrak{C}}$, the choice of $\omega \in \Omega_n$ is separately biassed by the numbers of components $C_1^{(n)}(\omega), \dots, C_n^{(n)}(\omega)$ of different sizes appearing in ω:

- If $\eta_i > 1$, an instance $\omega \in \Omega_n$ with a large number $C_i^{(n)}(\omega)$ of components of size i is more likely to be chosen than a structure with only few components of this size.

- If $0 < \eta_i < 1$ we have the opposite situation. An $\omega \in \Omega_n$ with small $C_i^{(n)}(\omega)$ is picked with higher probability than a structure with many components of this size i.
- If $\eta_i = 1$ there is no bias.
- Finally, if $\eta_i = 0$, any instance ω that has a component of size i is chosen with probability zero; it is not an element of $\widetilde{\Omega}_n$.

Remark 4.4. Clearly, every strictly positive real-valued sequence $H := \{\eta_i\}_{i\in\mathbb{N}}$ is admissible to tilt $\mathfrak{C}$. What is more, if $\eta_i := \eta > 0$ for each $i \in \mathbb{N}$, an instance $\omega \in \Omega_n$ is chosen with probability $\widetilde{\nu}_n(\{\omega\})$ proportional to

$$\eta^{\text{number of components in } \omega}.$$

This special case of H-tilting is examined by Arratia et al. (2003, Section 2.5).

4.1.3 The conditioning relation

Definition 4.5. The RDCS $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ satisfies the $\mathfrak{Z}$-*conditioning relation* for a sequence $\mathfrak{Z} := \{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables if there is an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ with $\Omega_n \neq \emptyset$ we have

$$\mathbb{P}\Big[\sum\nolimits_{i=1}^{n} iZ_i = n\Big] > 0\,. \tag{4.7}$$

and

$$\mathscr{L}_{\nu_n}\big(C^{(n)}\big) = \mathscr{L}\Big(Z_1, \ldots, Z_n \,\Big|\, \sum\nolimits_{i=1}^{n} iZ_i = n\Big)\,. \tag{4.8}$$

Lemma 4.6. *Assume that the RDCS $\mathfrak{C}$ satisfies the $\mathfrak{Z}$-conditioning relation for a sequence $\mathfrak{Z} := \{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables. Let $H := \{\eta_i\}_{i\in\mathbb{N}}$ be a non-negative sequence. We then consider the sequence $\widetilde{\mathfrak{Z}} := \{\widetilde{Z}_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables, where*

$$\mathbb{P}\big[\widetilde{Z}_i = k\big] := \frac{\eta_i^k}{\mathbb{E}\eta_i^{Z_i}}\mathbb{P}[Z_i = k] \qquad \textit{for all } k \in \mathbb{Z}_+, \tag{4.9}$$

if $\eta_i > 0$, and where

$$\mathbb{P}[\widetilde{Z}_i = 0] := 1 \tag{4.10}$$

in case that $\eta_i = 0$.

If H is admissible to tilt $\mathfrak{C}$, then the tilted structure $\widetilde{\mathfrak{C}} := \{(\widetilde{\Omega}_n, \widetilde{\nu}_n, \widetilde{C}^{(n)})\}_{n\in\mathbb{N}}$ satisfies the $\widetilde{\mathfrak{Z}}$-conditioning relation.

Proof. The proof is almost identical to the one for the constant case of Remark 4.4, which can be found in Arratia et al. (2003, Section 2.5). However, since we have to pay attention to the possibility that $\eta_i = 0$ can occur, the lemma is proved here nevertheless. To do so, let $n \geq n_0$ (with n_0 as in Definition 4.5) be such that $\widetilde{\Omega}_n \neq \emptyset$.

(i) We first show that

$$\mathbb{P}\Big[\sum\nolimits_{i=1}^n i\widetilde{Z}_i = n\Big] > 0\,.$$

By assumption, there is a $\omega_0 \in \widetilde{\Omega}_n \subset \Omega_n$, and we have $\nu_n(\{\omega_0\}) > 0$ from (4.1), and $\sum_{i=1}^n iC_i^{(n)}(\omega_0) = n$ from (4.2). Let $c_i := C_i^{(n)}(\omega_0)$ for all $1 \leq i \leq n$. The $\mathfrak{Z}$-conditioning relation for $\mathfrak{C}$ then yields

$$\frac{\mathbb{P}[Z_1 = c_1, \ldots, Z_n = c_n]}{\mathbb{P}[\sum_{i=1}^n iZ_i = n]} = \mathbb{P}_{\nu_n}\big[C_1^{(n)} = c_1, \ldots, C_n^{(n)} = c_n\big] > 0\,.$$

Invoking the independence of the Z_i, we conclude that

$$\prod_{\substack{i=1\\ \eta_i>0}}^n \mathbb{P}[Z_i = c_i] > 0\,,$$

which in turn implies that

$$\prod_{\substack{i=1\\ \eta_i>0}}^n \mathbb{P}[\widetilde{Z}_i = c_i] = \prod_{\substack{i=1\\ \eta_i>0}}^n \frac{\eta_i^{c_i}}{\mathbb{E}\eta_i^{Z_i}}\mathbb{P}[Z_i = c_i] > 0\,, \tag{4.11}$$

Because H is adimissible to tilt $\mathfrak{C}$, we also have

$$\frac{\prod_{i=1}^n \eta_i^{c_i}}{\mathbb{E}_{\nu_n}\Big\{\prod_{i=1}^n \eta_i^{C_i^{(n)}}\Big\}}\,\nu_n(\{\omega_0\}) = \widetilde{\nu}_n(\{\omega_0\}) > 0\,,$$

implying that $c_i = 0$ for every i with $\eta_i = 0$, so that $\sum_{i=1,\eta_i>0}^n ic_i = n$. It follows with (4.11) and the definition of $\mathbb{P}[\widetilde{Z}_i = 0] := 1$ that

$$\mathbb{P}\Big[\sum\nolimits_{i=1}^n i\widetilde{Z}_i = n\Big] \geq \prod_{i=1}^n \mathbb{P}[\widetilde{Z}_i = c_i] = \prod_{\substack{i=1\\ \eta_i>0}}^n \mathbb{P}[\widetilde{Z}_i = c_i] > 0\,.$$

(ii) We have also to show that

$$\mathbb{P}_{\widetilde{\nu}_n}\big[\widetilde{C}_1^{(n)} = c_1, \ldots, \widetilde{C}_n^{(n)} = c_n\big] = \mathbb{P}\Big[\widetilde{Z}_1 = c_1, \ldots, \widetilde{Z}_n = c_n \,\Big|\, \sum\nolimits_{i=1}^n i\widetilde{Z}_i = n\Big]$$

for all $c_1, \ldots, c_n \in \mathbb{Z}_+$.

If $\sum_{i=1}^n ic_i \neq n$, then (4.6) implies that

$$\mathbb{P}_{\widetilde{\nu}_n}\left[\widetilde{C}_1^{(n)} = c_1, \ldots, \widetilde{C}_n^{(n)} = c_n\right] = 0 \\ = \mathbb{P}\left[\widetilde{Z}_1 = c_1, \ldots, \widetilde{Z}_n = c_n \;\middle|\; \sum\nolimits_{i=1}^n i\widetilde{Z}_i = n\right]. \tag{4.12}$$

Now, assume that $\sum_{i=1}^n ic_i = n$. We define

$$I_n := \{1 \leq i \leq n : \eta_i > 0\} \qquad \text{and} \qquad I_n^{\mathrm{c}} := \{1 \leq i \leq n : \eta_i = 0\}.$$

If there is a $j \in I_n^{\mathrm{c}}$ such that $c_j > 0$, then the definition of $\widetilde{\nu}_n$ and the property that $\mathbb{P}[\widetilde{Z}_j = 0] = 1$ imply (4.12) as well.

It follows that the distributions $\mathscr{L}_{\widetilde{\nu}_n}(\widetilde{C}^{(n)})$ and $\mathscr{L}(\widetilde{Z}_1, \ldots, \widetilde{Z}_n | \sum_{i=1}^n i\widetilde{Z}_i = n)$ are concentrated on the set of vectors $(c_1, \cdots, c_n) \in \mathbb{Z}_+^n$ that satisfy $\sum_{i=1}^n ic_i = n$, and $c_j = 0$ for all $j \in I_n^{\mathrm{c}}$. Let $(c_1, \cdots, c_n)$ be such a vector. Recall the definition of ν_n^* in (4.5), and also recall that $\mathbb{P}[\sum_{i=1}^n iZ_i = n] > 0$ because $\Omega_n \neq \emptyset$ and $\mathfrak{C}$ satisfies the $\mathfrak{Z}$-conditioning relation. It follows that

$$\begin{aligned}
&\mathbb{P}_{\widetilde{\nu}_n}\left[\widetilde{C}_1^{(n)} = c_1, \ldots, \widetilde{C}_n^{(n)} = c_n\right] \\
&= \mathbb{P}_{\nu_n^*}\left[C_1^{(n)} = c_1, \ldots, C_n^{(n)} = c_n\right] \\
&= \frac{\prod_{i \in I_n} \eta_i^{c_i}}{\mathbb{E}_{\nu_n}\left\{\prod_{i=1}^n \eta_i^{C_i^{(n)}}\right\}} \mathbb{P}_{\nu_n}\left[C_1^{(n)} = c_1, \ldots, C_n^{(n)} = c_n\right] \\
&= \frac{\prod_{i \in I_n} \eta_i^{c_i}}{\mathbb{E}_{\nu_n}\left\{\prod_{i=1}^n \eta_i^{C_i^{(n)}}\right\}} \mathbb{P}\left[Z_1 = c_1, \ldots, Z_n = c_n \;\middle|\; \sum\nolimits_{i=1}^n iZ_i = n\right] \\
&= \frac{\prod_{i \in I_n} \eta_i^{c_i}}{\mathbb{E}_{\nu_n}\left\{\prod_{i=1}^n \eta_i^{C_i^{(n)}}\right\}} \frac{\prod_{i \in I_n} \mathbb{P}[Z_i = c_i] \prod_{i \in I_n^{\mathrm{c}}} \mathbb{P}[Z_i = 0]}{\mathbb{P}\left[\sum_{i=1}^n iZ_i = n\right]} \\
&= \prod_{i \in I_n} \frac{\eta_i^{c_i} \mathbb{P}[Z_i = c_i]}{\mathbb{E}\eta_i^{Z_i}} \frac{\prod_{i \in I_n} \mathbb{E}\eta_i^{Z_i} \prod_{i \in I_n^{\mathrm{c}}} \mathbb{P}[Z_i = 0]}{\mathbb{E}_{\nu_n}\left\{\prod_{i=1}^n \eta_i^{C_i^{(n)}}\right\} \mathbb{P}\left[\sum_{i=1}^n iZ_i = n\right]} \\
&= K_n(H)\, \mathbb{P}\left[\widetilde{Z}_1 = c_1, \ldots, \widetilde{Z}_n = c_n \;\middle|\; \sum\nolimits_{i=1}^n i\widetilde{Z}_i = n\right],
\end{aligned}$$

where

$$K_n(H) := \frac{\prod_{i \in I_n} \mathbb{E}\eta_i^{Z_i} \prod_{i \in I_n^{\mathrm{c}}} \mathbb{P}[Z_i = 0]}{\mathbb{E}_{\nu_n}\left\{\prod_{i=1}^n \eta_i^{C_i^{(n)}}\right\}} \frac{\mathbb{P}\left[\sum_{i=1}^n i\widetilde{Z}_i = n\right]}{\mathbb{P}\left[\sum_{i=1}^n iZ_i = n\right]}.$$

Summing up over all vectors $(c_1, \cdots, c_n)$ that satisfy $\sum_{i=1}^n ic_i = n$, and $c_j = 0$ for $j \in I_n^c$, yields $K_n(H) = 1$. $\square$

4.1.4 The "classical" RDCS

In this subsection we give a brief account of the "classical" types of RDCS; assemblies, which are labelled structures, and multisets and selections, which are unlabelled structures (see also Arratia et al. (2003, Section 2)).

4.1.4.1 Assemblies

Let $\{p(i)\}_{i\in\mathbb{N}}$ be a sequence of non-negative integers, and let

$$\big\{\mathrm{struct}_j(B) : B \subset \mathbb{N} \text{ non-empty and finite, and } 1 \le j \le p(|B|)\big\}$$

be a universe of pairwise distinct elements. An element $\mathrm{struct}_j(B)$ is interpreted an *additional structure* which the set B can be endowed with. The number of possible structures a set B can carry is $p(|B|)$. Note that $p(b) = 0$ means that all subsets $B \subset \mathbb{N}$ of size $|B| = b$ do not admit any additional structure. We define $\mathcal{P}_n$ to be the set of all partitions of $\{1, \cdots, n\}$ into pairwise disjoint non-empty sets, excluding those partitions which contain a set of a size that admits no additional structure. To construct an assembly of size n, we proceed as follows.

1. Let $n \in \mathbb{N}$ with $\mathcal{P}_n \neq \emptyset$. Choose uniformly one of the partitions in $\mathcal{P}_n$, say,
$$B_1 \cup \cdots \cup B_k = \{1, \cdots, n\}\,.$$
2. For each $1 \le j \le k$ we endow the set B_j randomly with an additional structure by choosing each of the $p(|B_j|)$ possibilities
$$\mathrm{struct}_1(B_j), \ldots, \mathrm{struct}_{p(|B_j|)}(B_j)$$
with equal probability. Assume that we have picked $\mathrm{struct}_{r_j}(B_j)$ for $1 \le j \le k$.
3. In this way we have constructed a set
$$\omega := \Big\{\big(B_1, \mathrm{struct}_{r_1}(B_1)\big), \ldots, \big(B_k, \mathrm{struct}_{r_k}(B_k)\big)\Big\}\,,$$
along with the probability of obtaining exactly this ω among all sets that are constructed in this way (for a given n). We denote this probability by $\nu_n(\{\omega\})$. Finally, we define $C_i^{(n)}(\omega)$ to be the number of partitions of size i underlying $\omega \in \Omega_n$.

Definition 4.7. Let $\mathcal{P}_n \neq \emptyset$ and let Ω_n be the (finite) set of all objects ω, constructed as in 1.–3. Let ν_n be the (uniform) probability measure on $(\Omega_n, \mathcal{P}(\Omega_n))$ induced by the probabilities $\nu_n(\{\omega\})$, $\omega \in \Omega_n$. By construction $\mathcal{C}^{(n)} := (\Omega_n, \nu_n, C^{(n)})$ is an RDCS of size n as in Definition 4.1, an *assembly of size* n associated with $\{p(i)\}_{i\in\mathbb{N}}$.

If $\mathcal{P}_n = \emptyset$ we set $\mathcal{C}^{(n)} := (\emptyset, 0, (0, \ldots, 0))$ as in Definition 4.2. The sequence $\mathfrak{C} := \{\mathcal{C}^{(n)}\}_{n\in\mathbb{N}}$ then is an RDCS. We call it an *assembly* associated with $\{p(i)\}_{i\in\mathbb{N}}$.

Lemma 4.8. *Let* $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ *be an assembly associated with the sequence* $\{p(i)\}_{i\in\mathbb{N}}$. *Let* $x > 0$, *and let* $\mathfrak{Z}(x) := \{Z_i(x)\}_{i\in\mathbb{N}}$ *be a sequence of independent Poisson distributed random variables,*

$$Z_i(x) \sim \mathrm{Po}(p(i)x^i/i!)\,, \qquad i \in \mathbb{N}\,, \tag{4.13}$$

where $\mathbb{P}[Z_i(x) = 0] = 1$ *if* $p(i) = 0$, *as stated in Appendix A.1.1. Then* $\mathfrak{C}$ *satisfies the* $\mathfrak{Z}(x)$*-conditioning relation.*

Proof. The proof of this lemma can be found in Arratia et al. (2003, Section 2.3). There, however, it is tacitly assumed that $p(i) \geq 1$ for all $i \in \mathbb{N}$. But if $p(i) = 0$ is allowed, the proof has to be extended only very little.

The first point is to note that, under our conditions, if $\Omega_n \neq \emptyset$ then also $\mathbb{P}[\sum_{i=1}^n iZ_i(x) = n] > 0$. Indeed, if $\Omega_n \neq \emptyset$, then there is an $\omega \in \Omega_n$, which is constructed with non-empty, finite, pairwise disjoint subsets of $\mathbb{N}$, $B_1, \ldots, B_k$, say. By construction, each of these sets admits an additional structure; that is, we have $p(|B_1|), \ldots, p(|B_k|) \geq 1$. Let c_i be the number of sets among $B_1, \ldots, B_k$ which have size i. We have $\sum_{i=1}^n ic_i = n$; and if $p(i) = 0$ then also $c_i = 0$. Hence, because the Poisson distribution of $Z_i(x)$ has positive mass on any $0, 1, \ldots, n$ if $p(i) \geq 1$,

$$\mathbb{P}\Big[\sum\nolimits_{i=1}^n iZ_i(x) = n\Big] \geq \prod_{i=1}^n \mathbb{P}[Z_i(x) = c_i] = \prod_{\substack{i=1\\p(i)\geq 1}}^n \mathbb{P}[Z_i(x) = c_i] > 0\,.$$

The second point is to note that if $p(i) = 0$ the fraction of instances $\omega \in \Omega_n$ that have no components of size i is equal to 1, which corresponds to $\mathbb{P}[Z_i(x) = 0] = 1$. $\square$

Lemma 4.9. *Let* $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ *be an assembly associated with the sequence* $\{p(i)\}_{i\in\mathbb{N}}$. *Let* $x > 0$, *and let* $\mathfrak{Z}(x) := \{Z_i(x)\}_{i\in\mathbb{N}}$ *be a sequence of independent Poisson random variables as in* (4.13).

Let $H := \{\eta_i\}_{i\in\mathbb{N}}$ be a non-negative sequence that is admissible to tilt $\mathfrak{C}$, and let $\widetilde{\mathfrak{Z}}(x) := \{\widetilde{Z}_i(x)\}_{i\in\mathbb{N}}$ be a sequence of independent Poisson random variables

$$\widetilde{Z}_i(x) \sim \mathrm{Po}(\eta_i p(i) x^i / i!)\,, \qquad i \in \mathbb{N}\,. \tag{4.14}$$

The H-tilted structure $\widetilde{\mathfrak{C}} := \{(\widetilde{\Omega}_n, \widetilde{\nu}_n, \widetilde{C}_n)\}_{n\in\mathbb{N}}$ satisfies the $\widetilde{\mathfrak{Z}}(x)$-conditioning relation.

Proof. If $Z_i(x) \sim \mathrm{Po}(p(i)x^i/i!)$ then the random variables $\widetilde{Z}_i(x)$, defined as in (4.9) and (4.10) of Lemma 4.6, satisfy $\widetilde{Z}_i(x) \sim \mathrm{Po}(\eta_i p(i) x^i/i!)$. Noting this, the lemma follows immediately from Lemma 4.8 and Lemma 4.6. □

4.1.4.2 Multisets

Let $\{p(i)\}_{i\in\mathbb{N}}$ be a sequence of non-negative integers, and let

$$\big\{\mathrm{object}_j(b) : b \in \mathbb{N} \text{ and } 1 \le j \le p(b)\big\}$$

be a universe of pairwise distinct elements. An element $\mathrm{object}_j(b)$ is interpreted an *object of size* b. In particular, $p(b) = 0$ means that there are no objects of size b in the universe. Let $\mathcal{P}_n$ be the set of all partitions of n into positive integers, excluding those partitions which contain an integer with no object of this size. To construct a multiset of size n, we proceed as follows.

1. Let $n \in \mathbb{N}$ with $\mathcal{P}_n \neq \emptyset$. Choose uniformly one of the partitions in $\mathcal{P}_n$, say
$$b_1 + \cdots + b_k = n\,.$$
2. For each $1 \le j \le k$ we choose uniformly one the $p(b_j)$ objects
$$\mathrm{object}_1(b_j), \ldots, \mathrm{object}_{p(b_j)}(b_j)\,.$$
Assume that we have picked $\mathrm{object}_{r_j}(b_j)$ for $1 \le j \le k$.
3. In this way we have constructed a multiset (in this context, a set whose elements need not be pairwise distinct)
$$\omega := \Big\{\big(b_1, \mathrm{object}_{r_1}(b_1)\big), \ldots, \big(b_k, \mathrm{object}_{r_k}(b_k)\big)\Big\}\,,$$
along with the probability of obtaining exactly this ω among all multisets that are constructed this way for the given n. This probability is denoted by $\nu_n(\{\omega\})$. Let $C_i^{(n)}(\omega)$ be the number of integers in the partition of n underlying ω that are equal to i.

Definition 4.10. Let $\mathcal{P}_n \neq \emptyset$. We define Ω_n to be the (finite) set of all objects ω, constructed as in 1.–3. Let ν_n be the (uniform) probability measure on $(\Omega_n, \mathcal{P}(\Omega_n))$ induced by the probabilities $\nu_n(\{\omega\})$, $\omega \in \Omega_n$. By construction, $\mathcal{C}^{(n)} := (\Omega_n, \nu_n, C^{(n)})$ is an RDCS of size n as in Definition 4.1, a *multiset of size* n associated with the sequence$\{p(i)\}_{i\in\mathbb{N}}$.

If $\mathcal{P}_n = \emptyset$ we set $\mathcal{C}^{(n)} := (\emptyset, 0, (0, \ldots, 0))$. The sequence $\mathfrak{C} := \{\mathcal{C}^{(n)}\}_{n\in\mathbb{N}}$ is an RDCS as in Definition 4.2, which we refer to as a *multiset* associated with the sequence $\{p(i)\}_{i\in\mathbb{N}}$.

Lemma 4.11. *Let* $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ *be a multiset associated with the sequence* $\{p(i)\}_{i\in\mathbb{N}}$*. Let* $0 < x < 1$*, and let* $\mathfrak{Z}(x) := \{Z_i(x)\}_{i\in\mathbb{N}}$ *be a sequence of independent negative binomially distributed random variables,*

$$Z_i(x) \sim \mathrm{NB}(p(i), x^i)\,, \qquad i \in \mathbb{N}\,, \tag{4.15}$$

where $\mathbb{P}[Z_i(x) = 0] = 1$ *if* $p(i) = 0$*, as stated in Appendix A.1.1. Then* $\mathfrak{C}$ *satisfies the* $\mathfrak{Z}(x)$*-conditioning relation.*

Proof. The proof can be found in Arratia et al. (2003, Section 2.3), at least if $p(i) > 0$ for all $i \in \mathbb{N}$. Under our assumptions, where $p(i) = 0$ can occur, the proof has to be slightly extended, much as in the proof of Lemma 4.8. As there, we have $\mathbb{P}[\sum_{i=1}^n iZ_i(x) = n] > 0$ if $\Omega_n \neq \emptyset$. Further, if $p(i) = 0$ the fraction of instances $\omega \in \Omega_n$ that have no components of size i is equal to 1, which corresponds to $\mathbb{P}[Z_i(x) = 0] = 1$. □

Lemma 4.12. *Let* $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ *be a multiset associated with the sequence* $\{p(i)\}_{i\in\mathbb{N}}$*. Let* $0 < x < 1$*. Let* $\mathfrak{Z}(x) := \{Z_i(x)\}_{i\in\mathbb{N}}$ *be a sequence of independent negative binomially distributed random variables as in* (4.15)*.*

Let $H := \{\eta_i\}_{i\in\mathbb{N}}$ *be a non-negative sequence that is admissible to tilt* $\mathfrak{C}$ *and that satisfies*

$$\eta_i x^i < 1 \qquad \text{for all } i \in \mathbb{N}.$$

Let $\widetilde{\mathfrak{Z}}(x) := \{\widetilde{Z}_i(x)\}_{i\in\mathbb{N}}$ *be a sequence of independent negative binomially distributed random variables*

$$\widetilde{Z}_i(x) \sim \mathrm{NB}(p(i), \eta_i x^i)\,, \qquad i \in \mathbb{N}\,. \tag{4.16}$$

The H*-tilted structure* $\widetilde{\mathfrak{C}} := \{(\widetilde{\Omega}_n, \widetilde{\nu}_n, \widetilde{C}_n)\}_{n\in\mathbb{N}}$ *satisfies the* $\tilde{\mathfrak{Z}}(x)$*-conditioning relation.*

Proof. We have $Z_i(x) \sim \mathrm{NB}(p(i), x^i)$ and $0 \leq \eta_i x^i < 1$. Then $\widetilde{Z}_i(x)$, defined as in Lemma 4.6, satisfies $\widetilde{Z}_i(x) \sim \mathrm{NB}(p(i), \eta_i x^i)$. But then the lemma is a consequence of Lemma 4.11 and Lemma 4.6. □

4.1.4.3 Selections

Selections are constructed similarly to multisets in Subsection 4.1.4.2. The only difference in the construction is that in step 2 we do not allow the same object to be chosen twice, so that the elements of ω in the third step now are pairwise distinct. As an analogue of Definition 4.10 we have:

Definition 4.13. Let $\mathcal{P}_n \neq \emptyset$. We define Ω_n to be the (finite) set of all objects ω. Let ν_n be the (uniform) probability measure on $(\Omega_n, \mathcal{P}(\Omega_n))$ induced by the probabilities $\nu_n(\{\omega\})$, $\omega \in \Omega_n$. By construction, $\mathcal{C}^{(n)} := (\Omega_n, \nu_n, C^{(n)})$ is an RDCS of size n, a *selection of size n* associated with $\{p(i)\}_{i\in\mathbb{N}}$.

If $\mathcal{P}_n = \emptyset$ we set $\mathcal{C}^{(n)} := (\emptyset, 0, (0, \dots, 0))$ and the sequence $\mathfrak{C} := \{\mathcal{C}^{(n)}\}_{n\in\mathbb{N}}$ is an RDCS, a *selection* associated with the sequence $\{p(i)\}_{i\in\mathbb{N}}$.

Lemma 4.11 and Lemma 4.12 hold mutatis mutandis in the context of selections. In particular, the negative binomial distributions of $Z_i(x)$ and $\widetilde{Z}_i(x)$ have to be replaced by binomial distributions

$$Z_i(x) \sim \mathrm{Bin}(p(i), x^i/(1+x^i)) \quad \text{and} \quad \widetilde{Z}_i(x) \sim \mathrm{Bin}(p(i), \eta_i x^i/(1+\eta_i x^i)),$$

where $x > 0$.

4.1.5 Hybrid ESF-structures

A fundamental example of an RDCS are *random permutations*. Random permutations are the assembly $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ associated with the sequence $\{p(i)\}_{i\in\mathbb{N}}$ with

$$p(i) := (i-1)! \qquad \text{for all } i \in \mathbb{N}.$$

For each finite set $B \subset \mathbb{N}$ of size b the additional structures are the $(b-1)!$ permutations of B that consist of exactly one cycle. Lemma 4.8 implies, choosing $x := 1$, that $\mathfrak{C}$ satisfies the $\mathfrak{Z}$-conditioning relation for the sequence $\mathfrak{Z} := \{Z_i\}_{i\in\mathbb{N}}$, where

$$Z_i \sim \mathrm{Po}(1/i) \qquad \text{for all } i \in \mathbb{N}. \tag{4.17}$$

We examine some RDCS which arise from the assembly of random permutations $\mathfrak{C}$ through tilting. We start with $H := \{\eta_i\}_{i\in\mathbb{N}}$, where $\eta_i := \eta > 0$, for all $i \in \mathbb{N}$. In this case the component spectra of the H-tilted structure $\widetilde{\mathfrak{C}} := \{(\widetilde{\Omega}_n, \widetilde{\nu}_n, \widetilde{C}^{(n)})\}_{n\in\mathbb{N}}$ satisfy

$$\mathscr{L}_{\widetilde{\nu}_n}\big(\widetilde{C}^{(n)}\big) = \mathscr{L}\Big(\widetilde{Z}_1, \dots, \widetilde{Z}_n \,\Big|\, \sum\nolimits_{i=1}^n i\widetilde{Z}_i = n\Big) \qquad \text{for all } n \in \mathbb{N},$$

with

$$\widetilde{Z}_i \sim \mathrm{Po}(\eta/i) \qquad \text{for all } i \in \mathbb{N}.$$

The distribution $\mathscr{L}_{\widetilde{\nu}_n}(\widetilde{C}^{(n)})$ is the *Ewens Sampling Formula* (Ewens, 1972), which we denote by $\mathrm{ESF}_n(\eta)$.

Inspired by Arratia et al. (1995, Section 4.4) we define a generalized version of $\mathrm{ESF}_n(\eta)$. For a non-negative sequence $H := \{\eta_i\}_{i\in\mathbb{N}}$ with integer period r, let

$$\mathrm{ESF}_n(\eta_1, \ldots, \eta_r) := \mathscr{L}\Big(\widetilde{Z}_1, \ldots, \widetilde{Z}_n \;\Big|\; \sum\nolimits_{i=1}^n i\widetilde{Z}_i = n\Big)$$

for all $n \in \mathbb{N}$ with $\mathbb{P}[\sum_{i=1}^n i\widetilde{Z}_i = n] > 0$, where

$$\widetilde{Z}_i \sim \mathrm{Po}(\eta_i/i) \qquad \text{for all } i \in \mathbb{N}.$$

Arratia et al. (1995) examined this distribution in the special case $r = 2$, with η_1 and η_2 strictly positive, and referred to it as a "hybrid" Ewens Sampling Formula. We adopt this notion for arbitrary r.

Definition 4.14. Let $H := \{\eta_i\}_{i\in\mathbb{N}}$ be a non-negative sequence with integer period r. An RDCS $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ that satisfies

$$\mathscr{L}_{\nu_n}\big(C^{(n)}\big) = \mathrm{ESF}_n(\eta_1, \ldots, \eta_r) \qquad \text{for all } n \text{ with } \Omega_n \neq \emptyset$$

is called a *hybrid ESF-structure*, or $\mathrm{ESF}(\eta_1, \ldots, \eta_r)$*-structure*.

If $\eta_1, \ldots, \eta_r \in \mathbb{Z}_+$, then a hybrid ESF-structure can be interpreted as an assembly of random permutations with coloured cycles using r palettes: cycles of length $1, r+1, 2r+1, \ldots$ are coloured using the η_1 colours from the first palette, cycles of length $2, r+2, 2r+2, \ldots$ are coloured using the η_2 colours from the second palette, and so on. If $\eta_i = 0$, there are no cycles of length i.

Example 4.15. The assembly $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ associated with the sequence $\{p(i)\}_{i\in\mathbb{N}}$, with

$$p(i) := \begin{cases} (i-1)! & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

is the assembly of random permutations without cycles of even size. We have

$$\mathscr{L}_{\nu_n}\big(C^{(n)}\big) = \mathrm{ESF}_n(1, 0) \qquad \text{for all } n \in \mathbb{N}.$$

If $\mathfrak{C}$ is associated with

$$p(i) := \begin{cases} (i-1)! & \text{if } i \text{ is odd,} \\ 2(i-1)! & \text{if } i \text{ is even,} \end{cases}$$

we have an assembly of random permutations with cycles of even size of two colours. Here

$$\mathscr{L}_{\nu_n}\big(C^{(n)}\big) = \mathrm{ESF}_n(1,2) \qquad \text{for all } n \in \mathbb{N}.$$

4.2 Quasi-logarithmic structures

4.2.1 Definition of a quasi-logarithmic RDCS

Quasi-logarithmic structures are defined by combining the notion of a quasi-logarithmic sequence of random variables, as introduced in Definition 2.32, with the conditioning relation from Definition 4.5. Recall from Theorem 3.21 that if $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{SUQLC}(\theta,\mathfrak{r})$ for some constant $\theta > 0$ and a positive integer sequence $\mathfrak{r}$, then

$$\lim_{n\to\infty} n\mathbb{P}\Big[\sum\nolimits_{i=1}^{n} iZ_i = n\Big] = p_\theta(1) > 0\,.$$

In particular, $\mathbb{P}[\sum_{i=1}^{n} iZ_i = n] > 0$ for all n large enough, for all $n \geq n_1$, say.

For this subsection, let $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ be an RDCS that, for some $n_2 \in \mathbb{N}$, satisfies $\Omega_n \neq \emptyset$ for all $n \geq n_2$.

If we assume that $\mathfrak{C}$ satisfies the $\mathfrak{Z}$-conditioning relation, $\mathfrak{Z} := \{Z_i\}_{i\in\mathbb{N}}$, it follows that for every $n \geq n_0 := n_1 \vee n_2$ we have

$$\mathscr{L}_{\nu_n}\big(C^{(n)}\big) = \mathscr{L}\Big(Z_1,\ldots,Z_n \,\Big|\, \sum\nolimits_{i=1}^{n} iZ_i = n\Big)\,.$$

Definition 4.16. Let $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ be an RDCS such that $\Omega_n \neq \emptyset$ for all n large enough. It is $(\theta,\mathfrak{r},\mathfrak{Z})$-*logarithmic* for a constant $\theta > 0$, a positive integer sequence $\mathfrak{r}$ and a sequence $\mathfrak{Z} := \{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables, if $\mathfrak{Z}$ satisfies condition $\mathsf{ULC}(\theta,\mathfrak{r})$ and if $\mathfrak{C}$ satisfies the $\mathfrak{Z}$-conditioning relation. The RDCS $\mathfrak{C}$ is *logarithmic* if there exist θ, $\mathfrak{r}$ and $\mathfrak{Z}$ as above, such that $\mathfrak{C}$ is $(\theta,\mathfrak{r},\mathfrak{Z})$-logarithmic.

This notion of a logarithmic RDCS is, essentially, used by Arratia et al. (2000) and Arratia et al. (2003) (cf. the discussion at the end of Subsection 2.1.1). Here, we extend this definition as follows.

Definition 4.17. Let $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ be an RDCS such that $\Omega_n \neq \emptyset$ for all n large enough. The structure is $(\theta,\mathfrak{r},\mathfrak{Z})$-*quasi-logarithmic* for a constant $\theta > 0$, a positive integer sequence $\mathfrak{r}$ and a sequence $\mathfrak{Z} := \{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables, if $\mathfrak{Z}$ satisfies condition $\mathsf{SUQLC}(\theta,\mathfrak{r})$ and if $\mathfrak{C}$ satisfies the $\mathfrak{Z}$-conditioning relation. The RDCS $\mathfrak{C}$ is *quasi-logarithmic* if there exist θ, $\mathfrak{r}$ and $\mathfrak{Z}$ as above, such that $\mathfrak{C}$ is $(\theta,\mathfrak{r},\mathfrak{Z})$-quasi-logarithmic.

Proposition 4.18. *Every $(\theta, \mathfrak{r}, \mathfrak{Z})$-logarithmic RDCS is $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic.*

Proof. This follows immediately from Proposition 2.33. $\square$

4.2.2 Examples

4.2.2.1 Assemblies

In this subsubsection let $\mathfrak{C}$ be an assembly associated with a sequence $\{p(i)\}_{i\in\mathbb{N}}$. Recall from Lemma 4.8 that $\mathfrak{C}$ satisfies, for any $x > 0$, the $\mathfrak{Z}(x)$-conditioning relation for the sequence $\mathfrak{Z}(x) := \{Z_i(x)\}_{i\in\mathbb{N}}$ of independent Poisson random variables, $Z_i(x) \sim \mathrm{Po}(p(i)x^i/i!)$.

Lemma 4.19. *The assembly $\mathfrak{C}$ is quasi-logarithmic if and only if there exist $\theta > 0$ and $x > 0$ such that*

$$\operatorname*{alim}_{i\to\infty} i\mathbb{E}Z_i(x) = \theta\,, \quad \textit{or, equivalently,} \quad \operatorname*{alim}_{i\to\infty} \frac{p(i)x^i}{(i-1)!} = \theta\,, \tag{4.18}$$

and where

$$\lim_{n\to\infty} d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum\nolimits_{i=b_n+1}^{n} iZ_i(x)\Big), \mathscr{L}\Big(\sum\nolimits_{i=b_n+1}^{n} iZ_i(x)+1\Big)\Big) = 0 \tag{4.19}$$

for every non-negative integer sequence $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} b_n/n = 0$. If so, the assembly $\mathfrak{C}$ is $(\theta, \mathfrak{r}, \mathfrak{Z}(x))$-quasi-logarithmic for every positive integer sequence $\mathfrak{r}$.

Proof. This follows immediately from Lemma 2.27. $\square$

Recalling Lemma 2.34, a simple condition on $\{p(i)\}_{i\in\mathbb{N}}$ in order that $\mathfrak{C}$ is quasi-logarithmic is given by $p(i) \asymp (i-1)!x^{-i}$, together with (4.18).

In view of Lemma 4.19, we refer to any $(\theta, \mathfrak{r}, \mathfrak{Z}(x))$-quasi-logarithmic assembly as (θ, x)-*quasi-logarithmic assembly* briefly.

Lemma 4.20. *Assume that $\mathfrak{C}$ is (θ, x)-quasi-logarithmic and (η, y)-quasi-logarithmic. Then $\theta = \eta$ and $x = y$.*

Proof. We assume that $x \le y$. The A-convergence in (4.18) particularly states that there is a positive integer sequence $\{m_n\}_{n\in\mathbb{N}}$, $m_n \to \infty$ and $m_n = \mathrm{o}(n)$, such that

$$\lim_{n\to\infty} \frac{1}{m_n} \sum_{i=1}^{m_n} \frac{p(i)x^i}{(i-1)!} = \theta \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{m_n} \sum_{i=1}^{m_n} \frac{p(i)y^i}{(i-1)!} = \eta\,.$$

Choose another positive integer sequence $\{l_n\}_{n\in\mathbb{N}}$, $l_n \to \infty$, such that

$$\lim_{n\to\infty} \frac{1}{m_n} \sum_{i=1}^{l_n} \frac{p(i)x^i}{(i-1)!} = 0\,.$$

Since $0 < x \le y$, we have

$$\frac{1}{m_n} \sum_{i=1}^{m_n} \frac{p(i)y^i}{(i-1)!} \ge \frac{1}{m_n} \sum_{i=l_n+1}^{m_n} \frac{p(i)x^i}{(i-1)!} \Big(\frac{y}{x}\Big)^{l_n} \qquad \text{for all } n \in \mathbb{N}.$$

The left hand side of the inequality converges to η as $n \to \infty$, whereas the right hand side converges to ∞ if $y/x > 1$. Therefore, we must have $x = y$. This in turn implies $\theta = \eta$. □

Recalling Lemma 2.10, a logarithmic assembly $\mathfrak{C}$ is characterized by the simple relation

$$\lim_{i\to\infty} i\mathbb{E}Z_i(x) = \theta\,, \quad \text{or, equivalently,} \quad \lim_{i\to\infty} \frac{p(i)x^i}{(i-1)!} = \theta\,, \tag{4.20}$$

for some $x > 0$ and $\theta > 0$. If so, we refer to $\mathfrak{C}$ as (θ, x)-*logarithmic assembly.*

Examples of logarithmic assemblies are *permutations* ($\theta = 1$, $x = 1$, cf. Subsection 4.1.5), 2-*regular graphs* ($\theta = 1/2$, $x = 1$) and *mappings* ($\theta = 1/2$, $x = 1/\mathrm{e}$). ESF(η)-structures are assemblies, as long as $\eta \in \mathbb{N}$ ($\theta = \eta$, $x = 1$). If $\eta \notin \mathbb{N}$, an ESF(η)-structure cannot be constructed as an assembly (cf. Subsubsection 4.1.4.1, where the $p(i)$ would not be integers anymore). Still, these structures satisfy the conditioning relation with Poisson distributed random variables, and are "close" to assemblies; see also Arratia et al. (2003, Section 2) for details.

Further, more exotic, examples of assemblies are so-called octopuses and children's yards, which we explain in the following two examples.

Example 4.21. The assembly associated to the sequence

$$p(i) := (i-1)!(2^i - 1) \qquad \text{for all } i \in \mathbb{N},$$

introduced by Bergeron (1990), is called the assembly of *octopuses* (cf. also Bergeron et al. (1998, p. 12)). Given a set of i elements, there are $(i-1)!$ possibilities to form a cycle with these elements. Then there are $2^i - 1$ possibilities to choose a non-empty subset of $\{1, \dots, i\}$. These elements form the head of the octopus, the remaining elements the tentacles. Octopuses form a logarithmic assembly; it satisfies (4.20) with $\theta = 1$ and $x = 1/2$.

The sequence

$$p(i) := (i-1)!\Big(\Big(\frac{1+\sqrt{5}}{2}\Big)^i + \Big(\frac{1-\sqrt{5}}{2}\Big)^i - (-1)^i - 1\Big) \qquad \text{for all } i \in \mathbb{N},$$

yields the assembly of octopuses without tentacles of even length (Bergeron et al., 1998, p. 56). It is logarithmic with $\theta = 1$ and $x = 2/(1+\sqrt{5})$.

Example 4.22. The assembly of *children's yards* is associated to the sequence

$$p(i) := i(i-2)! \qquad \text{for all } i \in \mathbb{N}.$$

For a set of size i there are i possibilities to pick an element as a child, and $(i-2)!$ possibilities to form a cycle around this child (cf. Flajolet and Soria (1990, Example 2)). This assembly is logarithmic with $\theta = 1$ and $x = 1$.

All of these examples are, because they are logarithmic, also quasi-logarithmic. A typical example of a quasi-logarithmic assembly that is not logarithmic is a hybrid $\mathrm{ESF}(\eta_1, \ldots, \eta_r)$-structure, where $\eta_1, \ldots, \eta_r \in \mathbb{N}$ are distinct (cf. Subsection 4.1.5). Permutations without cycles of even lengths ($\mathrm{ESF}_n(1,0)$-structures) are quasi-logarithmic assemblies that are not logarithmic (see Example 4.15).

4.2.2.2 Multisets

In this subsubsection let $\mathfrak{C}$ be a multiset associated with a sequence $\{p(i)\}_{i\in\mathbb{N}}$. Recall from Lemma 4.11 that $\mathfrak{C}$ satisfies, for any $0 < x < 1$, the $\mathfrak{Z}(x)$-conditioning relation for the sequence $\mathfrak{Z}(x) := \{Z_i(x)\}_{i\in\mathbb{N}}$ of independent negative binomially distributed random variables, $Z_i(x) \sim \mathrm{NB}(p(i), x^i)$.

Lemma 4.23. *The selection $\mathfrak{C}$ is quasi-logarithmic if and only if there exist $\theta > 0$ and $0 < x < 1$ such that*

$$\operatorname*{alim}_{i\to\infty} i\mathbb{E}Z_i(x) = \theta\,, \quad \textit{or, equivalently,} \quad \operatorname*{alim}_{i\to\infty} ip(i)x^i = \theta\,, \tag{4.21}$$

and where

$$\lim_{n\to\infty} d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum\nolimits_{i=b_n+1}^{n} iZ_i(x)\Big), \mathscr{L}\Big(\sum\nolimits_{i=b_n+1}^{n} iZ_i(x)+1\Big)\Big) = 0 \tag{4.22}$$

for every non-negative integer sequence $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} b_n/n = 0$. If so, the multiset $\mathfrak{C}$ is $(\theta, \{p(i)\}_{i\in\mathbb{N}}, \mathfrak{Z}(x))$-quasi-logarithmic.

Proof. This follows from Lemma 2.29. □

Note that a simple sufficient condition for $\mathfrak{C}$ to be quasi-logarithmic is given by (4.21) and $p(i) \asymp i^{-1}x^{-i}$ (cf. Lemma 2.34).

In view of Lemma 4.23, we refer to any $(\theta, \{p(i)\}_{i\in\mathbb{N}}, \mathfrak{Z}(x))$-quasi-logarithmic multiset as (θ, x)-*quasi-logarithmic multiset.*

Lemma 4.24. *Assume that $\mathfrak{C}$ is (θ, x)-quasi-logarithmic and (η, y)-quasi-logarithmic. Then $\theta = \eta$ and $x = y$.*

Proof. The proof is the same as that of Lemma 4.20. □

Recalling Lemma 2.10, a logarithmic multiset $\mathfrak{C}$ is characterized by

$$\lim_{i\to\infty} i\mathbb{E}Z_i(x) = \theta\,, \quad \text{or, equivalently,} \quad \lim_{i\to\infty} ip(i)x^i = \theta\,, \tag{4.23}$$

for some $x > 0$ and $\theta > 0$; $\mathfrak{C}$ is then called (θ, x)-*logarithmic multiset.*

Examples of logarithmic multisets are *mapping patterns* ($\theta = 1/2$, $x \approx 0.3383$) and *necklaces of aperiodic words* over an alphabet of size q ($\theta = 1$, $x = 1/q$). Note that necklaces of aperiodic words over an alphabet whose size q is a prime power correspond to *monic polynomials* in one indeterminate over a finite field with q elements (cf. Example 4.28 below). We refer to Arratia et al. (2003, Section 2) for details. As monic polynomials, many examples of logarithmic and quasi-logarithmic multisets arise from additive number systems, which we treat separately in Section 4.3 below.

4.2.2.3 Selections

In this subsubsection let $\mathfrak{C}$ be a selection associated with a sequence $\{p(i)\}_{i\in\mathbb{N}}$. Recall from Lemma 4.11 that $\mathfrak{C}$ satisfies, for any $x > 0$, the $\mathfrak{Z}(x)$-conditioning relation for the sequence $\mathfrak{Z}(x) := \{Z_i(x)\}_{i\in\mathbb{N}}$ of independent binomially distributed random variables, $Z_i(x) \sim \mathrm{Bin}(p(i), x^i(1 + x^i),)$.

Lemma 4.25. *The selection $\mathfrak{C}$ is quasi-logarithmic if and only if there exist $\theta > 0$ and $x > 0$ such that*

$$\operatorname*{alim}_{i\to\infty} i\mathbb{E}Z_i(x) = \theta\,, \quad \textit{or, equivalently,} \quad \operatorname*{alim}_{i\to\infty} ip(i)x^i = \theta\,, \tag{4.24}$$

and where

$$\lim_{n\to\infty} d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum\nolimits_{i=b_n+1}^{n} iZ_i(x)\Big), \mathscr{L}\Big(\sum\nolimits_{i=b_n+1}^{n} iZ_i(x) + 1\Big)\Big) = 0 \tag{4.25}$$

for every non-negative integer sequence $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} b_n/n = 0$. If so, the selection $\mathfrak{C}$ is $(\theta, \{p(i)\}_{i\in\mathbb{N}}, \mathfrak{Z}(x))$-quasi-logarithmic.

Proof. The lemma is a consequence of Lemma 2.28. □

As in the multiset case, a simple sufficient condition for $\mathfrak{C}$ to be quasi-logarithmic is given by (4.24) together with $p(i) \asymp i^{-1}x^{-i}$ (cf. Lemma 2.34).

We call a $(\theta, \{p(i)\}_{i\in\mathbb{N}}, \mathfrak{Z}(x))$-quasi-logarithmic selection a (θ, x)-*quasi-logarithmic selection.*

Lemma 4.26. *Assume that $\mathfrak{C}$ is (θ, x)-quasi-logarithmic and (η, y)-quasi-logarithmic. Then $\theta = \eta$ and $x = y$.*

Proof. The proof is the same as the proof of Lemma 4.20. □

A logarithmic selection $\mathfrak{C}$ is characterized by

$$\lim_{i\to\infty} i\mathbb{E}Z_i(x) = \theta\,, \quad \text{or, equivalently,} \quad \lim_{i\to\infty} ip(i)x^i = \theta\,, \tag{4.26}$$

for some $x > 0$ and $\theta > 0$ (cf. Lemma 2.10). Such a selection is called (θ, x)-*logarithmic selection.*

An example of a logarithmic selection is given by *square free monic polynomials* in one indeterminate over a finite field with q elements. We have $\theta = 1$ and $x = 1/q$ (Arratia et al., 2003, Example 2.13).

4.3 Additive arithmetic semigroups

4.3.1 Definition of an AAS

Recall that a *commutative monoid* is a triple $(A, \circ, e)$ where $(A, \circ)$ is a commutative semigroup and e is the neutral element for the operation $\circ$. An *additive norm* on $(A, \circ, e)$ is a mapping $\|\cdot\| : A \to \mathbb{Z}_+$ that satisfies

$$\|u\| = 0 \quad \Rightarrow \quad u = e \qquad \text{for all } u \in A,$$

and

$$\|u \circ v\| = \|u\| + \|v\| \qquad \text{for all } u, v \in A.$$

For every $B \subset A$ and every $n \in \mathbb{Z}_+$ we define

$$B(n) := \{u \in B : \|u\| = n\}\,,$$

the subset of all elements of B of norm n. This gives rise to a map

$$b : \mathbb{Z}_+ \to \mathbb{Z}_+ \cup \{\infty\}\,,$$

defined by

$$b(n) := \left|B(n)\right| \qquad \text{for all } n \in \mathbb{N},$$

the *counting function* of B. The generating series of B is denoted by

$$\mathbf{B}(z) := \sum_{n=0}^{\infty} b(n) z^n \,.$$

The counting function of A is denoted by $a(n)$, the generating series by $\mathbf{A}(z)$. Note that $a(0) = 1$, since $\|u\| = 0$ implies that $u = e$.

An element of $A \setminus \{e\}$ is called *indecomposable element* or *component* if it cannot be written as composition of two elements in $A \setminus \{e\}$. The set of all components of A is denoted by P. The counting function of P is denoted by $p(n)$; we refer to this function as the *component counting function*. The generating series of P is $\mathbf{P}(z)$.

The commutative monoid $(A, \circ, e)$ is *free* if every element of $A \setminus \{e\}$ can be expressed uniquely, up to associativity and commutativity, as a composition of indecomposable elements.

Definition 4.27. The tuple $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ is an *additive arithmetic semigroup (AAS)*, if $(A, \circ, e)$ is a free commutative monoid with a non-empty set $P \subset A$ of indecomposable elements, and if $\|\cdot\|$ an additive norm on $(A, \circ, e)$ such that the set $A(n) = \{u \in A : \|u\| = n\}$ is finite for each $n \in \mathbb{N}$.

Burris (2001) uses the terminology *additive number system* instead of additive arithmetic semigroup.

Although an AAS intends to mimic structural properties of the natural numbers and the primes, $\mathbb{N}$ itself does not carry a natural structure of an AAS because of the lack of a suitable additive norm that takes values in $\mathbb{Z}_+$.

The *radius of convergence* ρ of $\mathcal{A}$ is the radius of convergence of the generating series $\mathbf{A}(z) = \sum_{n=0}^{\infty} a(n) z^n$ of A. Because P is non-empty, $a(n) \geq 1$ holds true for infinitely many n, and we have $0 \leq \rho \leq 1$. An AAS $\mathcal{A}$ is called *reduced* if $\gcd\{n \in \mathbb{N} : p(n) > 0\} = 1$. Burris (2001, Lemma 2.42 and Theorem 2.52) shows that $\mathcal{A}$ is reduced if an only if $a(n) > 0$ for all n large enough.

If $0 \leq z < 1$ the so-called *Euler product formula* holds true, that is,

$$\sum_{n=0}^{\infty} a(n) z^n = \prod_{i=1}^{\infty} (1 - z^i)^{-p(i)} \,, \tag{4.27}$$

with the possibility that both the left and right hand side of (4.27) have the value ∞ (see Knopfmacher and Zhang (2001, p. 17ff) or Burris (2001, Corollary 2.24)).

Example 4.28. Let $(A, \circ, e)$ be the commutative monoid of monic polynomials in one indeterminate X over a finite field $\mathbb{F}_q$ with q elements, $\circ$ being the usual mutiplication and $e = 1$. Let $P \subset A$ be the subset of polynomials that are irreducible over $\mathbb{F}_q$. Let $\|f(X)\|$ be the degree of the poynomial $f(X) \in A$. Then $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ is a reduced AAS. Moreover, for $n \in \mathbb{N}$, we have

$$a(n) = q^n \qquad \text{and} \qquad p(n) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d} \sim \frac{q^n}{n} \, ,$$

with

$$\mu(d) := \begin{cases} 1 & \text{if } d = 1, \\ (-1)^r & \text{if } d \text{ is a product of } r \text{ distinct primes}, \\ 0 & \text{else}, \end{cases}$$

denoting the Möbius function; we refer to Knopfmacher and Zhang (2001, p. 74) for details. In particular, the radius of convergence of $\mathcal{A}$ is $1/q$.

There is a close relation between AAS and multisets. Every AAS $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ gives rise naturally to a multiset associated with the component counting function $p(i)$, $i \in \mathbb{N}$. Indeed, if $A(n) = \{\omega \in A : \|\omega\| = n\} \neq \emptyset$, we endow $A(n)$ with the uniform distribution ν_n, and define $C_i^{(n)}(\omega)$ be the number of indecomposable elements of norm i in the unique decomposition of $\omega \in A(n)$. Then $\mathcal{C}_{\mathcal{A}}^{(n)} := (A(n), \nu_n, C^{(n)})$ is a multiset of size n associated with $\{p(i)\}_{i \in \mathbb{N}}$, as constructed in Subsubsection 4.1.4.2. If $A(n) = \emptyset$, then we set $\mathcal{C}_{\mathcal{A}}^{(n)} := (\emptyset, 0, (0, \ldots, 0))$, and $\mathfrak{C}_{\mathcal{A}} := \{(A(n), \nu_n, C^{(n)})\}_{n \in \mathbb{N}}$ is a multiset associated with $\{p(i)\}_{i \in \mathbb{N}}$.

Definition 4.29. An AAS $\mathcal{A}$ is (θ, x)-*quasi-logarithmic* for $\theta > 0$ and $0 < x < 1$ if the associated multiset $\mathfrak{C}_{\mathcal{A}}$ is (θ, x)-quasi-logarithmic; $\mathcal{A}$ is (θ, x)-*logarithmic* if $\mathfrak{C}_{\mathcal{A}}$ is (θ, x)-logarithmic.

Remark 4.30. If an AAS $\mathcal{A}$ is quasi-logarithmic it is also reduced. This follows because the associated quasi-logarithmic multiset $\mathfrak{C}_{\mathcal{A}} := \{(A(n), \nu_n, C^{(n)})\}_{n \in \mathbb{N}}$ satisfies $A(n) \neq \emptyset$ for all n large enough (cf. Definition 4.17). Thus, $a(n) > 0$ for all n large enough as well.

4.3.2 Prime element theorems and quasi-logarithmic AAS

One aim in the theory of arithmetic semigroups is to translate results from number theory into the context of an AAS. Recall, for example, that the prime number theorem states that

$$\pi(x) \sim \frac{x}{\log x} \, ,$$

where $\pi(x)$ denotes the number of primes smaller or equal to $x \in \mathbb{R}$ (Tenenbaum, 1995, p. 10). The prime number theorem is thus a statement about the asymptotic behaviour of the proportion $\pi(n)/n$ of the number of primes among the first n natural numbers as $n \to \infty$.

In the context of an AAS $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ with counting functions $a(n)$ and $p(n)$, an analogous statement would consider the asymptotic behaviour of the proportion $p(n)/a(n)$ as $n \to \infty$. For monic polynomials over a finite field with q elements, as considered in Example 4.28, we have

$$a(n) = q^n \qquad \text{and} \qquad p(n) \sim \frac{q^n}{n} .$$

For a general AAS $\mathcal{A}$ a prime element theorem is a statement about the asymptotic behaviour of $p(n)$ as $n \to \infty$, given that the asymptotic behaviour of $a(n)$ as $n \to \infty$ is known:

$$\text{asymptotic behaviour of } a(n) \qquad \Rightarrow \qquad \text{asymptotic behaviour of } p(n).$$

Conversly, with an inverse prime element theorem we derive the asymptotic behaviour of $a(n)$ as $n \to \infty$ from the known asymptotics of the component counting function $p(n)$ as $n \to \infty$:

$$\text{asymptotic behaviour of } p(n) \qquad \Rightarrow \qquad \text{asymptotic behaviour of } a(n).$$

Our main interest lies in the interplay of (inverse) prime element theorems and the quasi-logarithmic condition. We show in this subsection that, invoking a prime element theorem of Zhang (1996a), an AAS $\mathcal{A}$ is quasi-logarithmic if the counting function $a(n)$ satisfies an analogue of a condition which Beurling (1937) used in the context of so-called generalized integers. In Subsection 4.3.3 we prove an inverse prime element theorem for quasi-logarithmic AAS, extending previous results of Knopfmacher and Warlimont (2002) and Arratia et al. (2003, 2005).

The asymptotic behaviour of $p(n)$ as $n \to \infty$ can be studied under different assumptions on the counting function $a(n)$. The assumptions on $a(n)$ have the form

$$a(n) = Q(n)q^n + R(n) \qquad \text{for all } n \in \mathbb{N}, \tag{4.28}$$

where $q > 1$ is a constant $Q(n)$ is a polynomial function and $R(n)$ is a remainder term that satisfies $\lim_{n\to\infty} R(n)/q^n = 0$. The classic case is *Knopfmacher's Axiom $A^\sharp$* (Knopfmacher, 1979), where

$$Q(n) = c \qquad \text{and} \qquad R(n) = \mathrm{O}\big(q^{\nu n}\big) , \tag{4.29}$$

for $c > 0$ and $0 \le \nu < 1$. The assumption on the remainder term $R(n)$ can be relaxed, and one can also consider

$$Q(n) = c \qquad \text{and} \qquad R(n) = \mathrm{O}\bigl(q^n n^{-\delta}\bigr)\,, \tag{4.30}$$

for $c > 0$ and $\delta > 1$. Wehmeier (2004) examines remainder terms of the form $R(n) = \mathrm{O}(q^n (\log n)^{-k})$ for all $k \in \mathbb{N}$.

As for $Q(n)$, Zhang (1996a, 1996b) considers more generally

$$Q(n) = \sum_{j=1}^{r} c_j n^{\rho_j - 1} \qquad \text{and} \qquad R(n) = \mathrm{O}\bigl(q^n n^{-\delta}\bigr)\,, \tag{4.31}$$

for $r \in \mathbb{N}$, real numbers $\rho_1 < \cdots < \rho_r$ and $c_1, \ldots, c_r$ such that $\rho_r > 0$ and $c_r > 0$, $q > 1$ and $\delta > 1$. Condition (4.31) is an analogue of the condition under which Beurling (1937) established prime number theorems of so-called *Beurling-generalized integers*; see Knopfmacher and Zhang (2001, Section 5.5) for details.

If $\mathcal{A}$ is an AAS where (4.28) and (4.30) hold with $\delta > 0$, it follows that

$$np(n)q^{-n} = 1 + \mathrm{o}(1) \qquad \text{if } \mathbf{A}(z) \neq 0 \text{ for } z = -1/q, \tag{4.32}$$

$$np(n)q^{-n} = 1 + (-1)^{n+1} + \mathrm{o}(1) \qquad \text{if } \mathbf{A}(z) = 0 \text{ for } z = -1/q, \tag{4.33}$$

where $\mathbf{A}(z) := \sum_{n=0}^{\infty} a(n) z^n$ (Knopfmacher and Zhang, 2001, Theorem 5.4.1). Note that the *non-classical* prime number theorem (4.33) was first obtained by Indlekofer et al. (1991), under the stronger condition (4.28).

Clearly, we have $np(n)q^{-n} = 1 + \mathrm{o}(1)$ if and only if $\mathcal{A}$ is $(1, 1/q)$-logarithmic (cf. (4.23) with $x := 1/q$ and $\theta := 1$). There are many examples of additive arithmetic semigroups that satisfy this form of prime element theorem, which is also called the *classic* abstract prime element theorem, and therefore provide examples of $(1, 1/q)$-logarithmic multisets. Beside the prototype example of monic polynomials in one indeterminate X over a finite field $\mathbb{F}_q$, there are finite modules over $\mathbb{F}_q[X]$, semisimple finite algebras over $\mathbb{F}_q[X]$, associate-classes of homogenous polynomials in $\mathbb{F}_q[X_1, X_2]$, and more. We refer to Knopfmacher and Zhang (2001, Subsection 3.1.1) for a comprehensive list of such algebraic examples. See also Knopfmacher and Warlimont (1999, 2002) for examples arising from asymptotic isomer enumeration in chemistry and the theory of maps on surfaces.

If $np(n)q^{-n} = 1 + (-1)^{n+1} + \mathrm{o}(1)$, then $\mathcal{A}$ is a $(1/2, 1/q)$-quasi-logarithmic AAS which is not logarithmic (cf. Example 2.37).

Remark 4.31. Knopfmacher and Zhang (2001, Examples 3.8.1 and 3.8.6) construct two purely analytical examples of AAS. The first satisfies

$$p(n) = \begin{cases} 2q^n n^{-1} + \mathrm{O}(1) & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases} \qquad \text{for some } q > 1,$$

the second

$$p(n) = q^n n^{-1} + \mathrm{O}(1) \qquad \text{for some } q > 1.$$

Both are examples of $(1, 1/q)$-quasi-logarithmic AAS, for the same reasons as Example 2.37.

The main theorem of this subsection connects the general condition (4.31) on $a(n)$ with the quasi-logarithmic condition.

Theorem 4.32. *Let $\mathcal{A}$ be an AAS that satisfies* (4.28) *and* (4.31) *with one of the following restrictions:*
(i) $0 < \rho_r < 1$ *and* $\delta > \max\{3/2, 1 + \rho_r\}$*,*
(ii) $\rho_r \geq 1$ *and* $\delta > 1 + \rho_r$*,*
(iii) $\rho_r \geq 1$*,* $\delta > 1 + \rho_r/2$ *and* $\mathbf{A}(-1/q) \neq 0$*.*
Then $\mathcal{A}$ is $(\rho_r, 1/q)$-quasi-logarithmic.

Proof. We show that the multiset $\mathfrak{C}_{\mathcal{A}}$ associated to $\mathcal{A}$ satisfies (4.21) and (4.22) with $\theta = \rho_r$ and $x = 1/q$ (recall that $q > 1$ under (4.28)). The argument of the proof relies on the abstract prime number theorems proved in Zhang (1996a).

First, we give a proof under assumption (ii).

(a) Assume that the generating series $\mathbf{A}(z)$ is zero at $z = -1/q$. In this case Zhang (1996a, Theorem 6.2) shows that there is a non-negative integer $s \leq (1 + \rho_r)/2$, there are positive integers $l_1, \ldots, l_s$ with

$$l_s + 2 \sum_{k=1}^{s-1} l_k \leq \rho_r\,, \tag{4.34}$$

and there are real numbers $0 < \omega_1 < \cdots < \omega_{s-1} < 1/2$, such that

$$\bar{\Lambda}(i) q^{-i} = \rho_r - 2 \sum_{k=1}^{s-1} l_k \cos(2\pi\omega_k i) - (-1)^i l_s + \mathrm{o}(1) \qquad \text{for all } n \in \mathbb{N}, \tag{4.35}$$

where $\bar{\Lambda}(i) := \sum_{j|i} jp(j)$. But $ip(i)q^{-i} = \bar{\Lambda}(i)q^{-i} + \mathrm{o}(1)$ if $a(n) = \mathrm{O}(q^n n^\alpha)$ for some $\alpha > 0$ (Knopfmacher and Zhang, 2001, Proposition 3.1.7), which is the case under (4.28) and (4.31). Therefore, we can write (4.35) in the form

$$ip(i)q^{-i} = \underbrace{\rho_r - 2 \sum_{k=1}^{s-1} l_k \cos(2\pi\omega_k i) - (-1)^i l_s}_{u_i} + v_i\,, \tag{4.36}$$

with $v_i \to 0$ is suitably chosen. The sequence $\{ip(i)q^{-i}\}_{i\in\mathbb{N}}$ is the sum of the integer skeleton $\{u_i\}_{i\in\mathbb{N}}$ of a sinusoidal function as described in Lemma 2.22 and a sequence $\{v_i\}_{i\in\mathbb{N}}$ that converges to 0. Lemma 2.22 implies that $\{u_i\}_{i\in\mathbb{N}}$ is $\mathsf{A}(\rho_r)$-convergent. Moreover, $\{v_i\}_{i\in\mathbb{N}}$ is $\mathsf{A}(0)$-convergent by Corollary 2.14. Then $\{ip(i)q^{-i}\}_{i\in\mathbb{N}}$ is $\mathsf{A}(\rho_r)$-convergent by Lemma 2.12; in other words, (4.21) is satisfied with $x = 1/q$.

We turn to the verification of (4.22). If $s = 0$ then (4.36) reduces to $ip(i)q^{-i} = \rho_r + v_i$. Because $v_i \to 0$, we have $ip(i)q^{-i} \geq \rho_r/2 > 0$ for all $i \geq n_1$, where $n_1 \in \mathbb{N}$ is large enough. If $s \geq 1$, (4.36) yields

$$ip(i)q^{-i} \geq \rho_r - 2\sum_{k=1}^{s-1} l_k - (-1)^i l_s + v_i \qquad \text{for all } i \in \mathbb{N}. \tag{4.37}$$

Choose $n_2 \in \mathbb{N}$ such that $v_i \geq -l_s/2$ for all $i \geq n_2$. We conclude from (4.37) and (4.34) that $ip(i)q^{-i} \geq l_s/2$ for each *odd* $i \geq n_2$.

Since $\{ip(i)q^{-i}\}_{i\in\mathbb{N}}$ is non-negative and $\mathsf{A}(\rho_r)$-convergent the sequence is bounded, by Lemma 2.18. We fix can an $0 < \varepsilon < 1$ and choose an $n_3 \in \mathbb{N}$ such that $1 - p(i)q^{-i} > \varepsilon$ for all $i \geq n_3$.

Recall from Lemma 4.11 that the multiset $\mathfrak{C}_{\mathcal{A}}$ associated with $\mathcal{A}$ particularly satisfies the $\mathfrak{Z}(1/q)$-conditioning relation with the sequence $\mathfrak{Z}(1/q) := \{Z_i(1/q)\}_{i\in\mathbb{N}}$ of independent negative binomially distributed random variables, $Z_i(1/q) \sim \mathrm{NB}(p(i), q^{-i})$. The Bernoulli inequality entails for each odd $i \geq \max\{n_1, n_3, n_3\}$ that

$$\begin{aligned} \mathbb{P}[Z_i(1/q) = 1] &= p(i)q^{-i}(1 - q^{-i})^{p(i)} \geq p(i)q^{-i}\big(1 - p(i)q^{-i}\big) \geq \frac{\varepsilon c_s}{2i}\,, \\ \mathbb{P}[Z_i(1/q) = 0] &= (1 - q^{-i})^{p(i)} \geq 1 - p(i)q^{-i} \geq \varepsilon\,, \end{aligned} \tag{4.38}$$

where $c_s := \rho_r$ if $s = 0$ and $c_s := l_s$ if $s \geq 0$. Theorem 6.27 now implies (4.22).

(b) Let $\mathbf{A}(z)$ be zero at $z = -1/q$. The argument is very similar to the one in (a). Here it follows from Zhang (1996a, Theorem 6.2) and Knopfmacher and Zhang (2001, Proposition 3.1.7) that there is a non-negative integer $s \leq (1 + \rho_r)/2$, there are positive integers $l_1, \ldots, l_s$ with

$$2\sum_{k=1}^{s} l_k \leq \rho_r\,, \tag{4.39}$$

and there are real numbers $0 < \omega_1 < \cdots < \omega_s < 1/2$ such that

$$ip(i)q^{-i} = \underbrace{\rho_r - 2\sum_{k=1}^{s} l_k \cos(2\pi\omega_k i)}_{u_i} + v_i\,, \tag{4.40}$$

where $v_i \to 0$. With the same argument as in (a) we conclude that $\{ip(i)q^{-i}\}_{i\in\mathbb{N}}$ is $\mathsf{A}(\rho_r)$-convergent and thus that (4.21) is satisfied with $x = 1/q$.

To prove (4.22), we consider the sequence $\mathfrak{Z}(1/q)$ of independent negative binomially distributed random variables as in (a). If $s = 0$, we have (4.40) is simply $ip(i)q^{-i} = \rho_r + v_i$. We deduce (4.22) from Theorem 6.27 as in (a). If, however, $s \geq 1$, Theorem 6.27 cannot be applied, because inequalities as in (4.38) cannot be obtained for *every* large enough odd i.

Still, we can infer from (4.40) that

$$i\mathbb{E}Z_i(1/q) = \frac{ip(i)q^{-i}}{1-q^{-i}} = u_i + v_i + q^{-i}\frac{ip(i)q^{-i}}{1-q^{-i}} \qquad \text{for all } i \in \mathbb{N}. \tag{4.41}$$

The last summand converges to 0 as $i \to \infty$, because $q > 0$ and $\{ip(i)q^{-i}\}_{i\in\mathbb{N}}$ is bounded under A-convergence. Combining (4.41) with (4.39) we see that the conditions of Theorem 6.18 are satisfied, and we conclude that (4.22) holds true. This proves the theorem under assumption (ii).

Under assumption (i), Zhang (1996a, Theorem 6.1) shows that $ip(i)q^{-i} \sim \rho_r$; $\mathcal{A}$ is $(\rho_r, 1/q)$-logarithmic in this case. Under assumption (iii) we have a similar situation as in (b) above, the only difference being $s \leq \rho_r/2$ instead of $s \leq (1 + \rho_r)/2$ (Zhang, 1996a, Theorem 6.2). □

4.3.3 An inverse prime element theorem for quasi-logarithmic AAS

We prove an inverse prime element theorem for a (θ, x)-quasi-logarithmic AAS $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$. That is, we derive the asymptotic behaviour of the counting function $a(n)$ of A from the behaviour of the component counting function $p(n)$ of P, which, under the assumption that $\mathcal{A}$ is (θ, x)-quasi-logarithmic, satisfies (4.21) and (4.22).

Theorem 4.33. *Let $\mathcal{A}$ be a (θ, x)-quasi-logarithmic AAS for some $\theta > 0$ and $0 < x < 1$. Then it follows that*

$$a(n) \sim c x^{-n} n^{\theta-1} \ell(n)\,, \tag{4.42}$$

where

$$0 < c := \frac{1}{\Gamma(\theta)}\left(\lim_{n\to\infty}\prod_{i=1}^{n}\left(\mathrm{e}^{x^i}(1-x^i)\right)^{-p(i)}\right) < \infty\,, \tag{4.43}$$

Γ being the Gamma function, and where

$$\ell(n) := \exp\left(\sum_{i=1}^{n}\frac{ip(i)x^i - \theta}{i}\right), \qquad n \in \mathbb{N} \tag{4.44}$$

is slowly varying at infinity.

Corollary 4.34. *Let $\mathcal{A}$ be an AAS. Then we have for all $\theta > 0$ and $0 < x < 1$*

$$p(n) \sim \theta x^{-n} n^{-1} \quad \Rightarrow \quad a(n) \sim c x^{-n} n^{\theta - 1} \ell(n)\,,$$

with $c > 0$ as in (4.43) and $\ell(n)$ as in (4.44).

Proof. Under the assumption on the component counting function $p(n)$, $\mathcal{A}$ is (θ, x)-logarithmic, and the assertion follows from Theorem 4.33. $\square$

Remark 4.35. Let $\theta > 0$ and $0 < x < 1$. Let $f : \mathbb{N} \to \mathbb{R}$ be a slowly varying function. Then there is (θ, x)-logarithmic AAS with a component counting function $p(n)$ such that $\ell(n)$, defined in (4.44), satisfies

$$\ell(n) \sim D f(n) \qquad \text{for some constant } D > 0. \tag{4.45}$$

That is, for any possible asymptotic growth behaviour that slowly varying functions allow, there is a (θ, x)-logarithmic AAS such that the $\ell(n)$ of (4.44) exhibits this asymptotic behaviour.

To show this, we recall Theorem A.2, and we assume that the sequences $\delta_i \to 0$ and $C_n \to C > 0$ represent $f(n)$ in the form

$$f(n) = C_n \exp\left(\sum_{i=1}^{n} \frac{\delta_i}{i}\right) \qquad \text{for all } n \in \mathbb{N}.$$

Now define

$$p(i) := \left\lfloor \frac{\theta + \delta_i}{i x^i} \right\rfloor = \frac{\theta + \delta_i}{i x^i} + \varepsilon_i \qquad \text{for all } i \in \mathbb{N},$$

where $\{\varepsilon_i\}_{i \in \mathbb{N}}$ is a suitable sequence in $[0, 1)$, making $\{p(i)\}_{i \in \mathbb{N}}$ integer valued. In particular, $\lim_{i \to \infty} i p(i) x^i = \theta$, so that $\mathcal{A}$ is (θ, x)-logarithmic. It follows that

$$\ell(n) = C'_n \exp\left(\sum_{i=1}^{n} \frac{\delta_i}{i}\right) \quad \text{with} \quad C'_n := \exp\left(\sum_{i=1}^{n} \varepsilon_i x^i\right) \qquad \text{for all } n \in \mathbb{N},$$

where $\{C'_n\}_{n \in \mathbb{N}}$ converges to some positive constant C' as $n \to \infty$, because $0 < x < 1$. But then (4.45) holds with $D := C'/C$.

Before we prove Theorem 4.33, we first recall from Lemma 4.11 that the mutltiset $\mathfrak{C}_{\mathcal{A}}$ associated to an AAS $\mathcal{A}$, with component counting function $p(n)$, satisfies the $\mathfrak{Z}(x)$-conditioning relation for any $0 < x < 1$ with a sequence $\mathfrak{Z}(x) := \{Z_i(x)\}_{i \in \mathbb{N}}$ of independent negative binomially distributed random variables, $Z_i(x) \sim \mathrm{NB}(p(i), x^i)$.

We introduce a new sequence $\{Z_i^*(x)\}_{i\in\mathbb{N}}$ of independent random variables as follows. Let $Z_i^*(x) := Z_i(x)$ if $p(i) > 0$, and let $Z_i^*(x)$ be negative binomially $\mathrm{NB}(1, x^i)$-distributed if $p(i) = 0$. We define

$$T_n(x) := \sum_{i=1}^{n} iZ_i(x) \quad \text{and} \quad T_n^*(x) := \sum_{i=1}^{n-1} iZ_i(x) + nZ_n^*(x) \qquad \text{for each } n \in \mathbb{N}. \tag{4.46}$$

The next result is an auxiliary lemma which is used in the proof of Theorem 4.33 below.

Lemma 4.36. *Let $\mathcal{A}$ be an AAS. Let $n \in \mathbb{N}$ be such that $a(n) > 0$, and let $0 < x < 1$. If $\mathbb{P}[T_n(x) = n] > 0$, we have*

$$a(n) = \frac{\mathbb{P}[T_n^*(x) = n]}{x^n(1-x^n)^{p(n)\vee 1}} \prod_{i=1}^{n-1} (1-x^i)^{-p(i)} - \delta_n\,, \tag{4.47}$$

where $\delta_n := \mathbf{1}\{p(n) = 0\}$.

Proof. Fix $n \in \mathbb{N}$ with $a(n) > 0$, and $0 < x < 1$. Let $\mathcal{C}_{\mathcal{A}}^{(n)} := (A(n), \nu_n, C^{(n)})$ be the multiset of size n associated with $\mathcal{A}$. Assume that $\mathbb{P}[T_n(x) = n] > 0$. We distinguish two cases, $p(n) > 0$ and $p(n) = 0$.

(i) First, assume that $p(n) > 0$. Here, we use the argument of Arratia et al. (2003, p. 223). Lemma 4.11 and the independence of the random variables $Z_1(x), \ldots, Z_n(x)$ yield

$$\begin{aligned}
\frac{p(n)}{a(n)} &= \mathbb{P}_{\nu_n}[C_n^{(n)} = 1] \\
&= \mathbb{P}\big[(Z_1(x), \ldots, Z_{n-1}(x), Z_n(x)) = (0, \ldots, 0, 1) \bigm| T_n(x) = n\big] \\
&= \frac{\mathbb{P}[Z_n(x) = 1]}{\mathbb{P}[T_n(x) = n]} \prod_{i=1}^{n-1} \mathbb{P}[Z_i(x) = 0] \\
&= \frac{p(n)x^n(1-x^n)^{p(n)}}{\mathbb{P}[T_n(x) = n]} \prod_{i=1}^{n-1} (1-x^i)^{p(i)}\,,
\end{aligned}$$

and this implies equation (4.47), because $T_n^*(x) = T_n(x)$ and $\delta_n = 0$.

(ii) Now, assume that $p(n) = 0$. We define a new AAS, denoted by $\mathcal{A}^* := (A^*, \circ^*, e^*, P^*, \|\cdot\|^*)$ say, as the direct sum of $\mathcal{A}$ and an auxiliary AAS that contains only one indecomposable element, which has norm n (cf. Burris (2001, p. 83)). Let $a^*(n)$ be the counting function of A^*, and $p^*(n)$ that of P^*. By construction we have $a^*(n) = a(n) + 1$, $p^*(i) = p(i)$ for $i \neq n$, and, $p^*(n) =$

$p(n) + 1 = 1$. We denote by $\mathfrak{C}_{\mathcal{A}^*} := \{(A^*(m), \nu^*_m, C^{(m)*})\}_{m\in\mathbb{N}}$ the multiset associated with $\mathcal{A}^*$. For our fixed n, $\mathbb{P}[T^*_n(x) = n] \geq \mathbb{P}[Z^*_n(x) = 1] > 0$ because $Z^*_n(x) \sim \mathrm{NB}(1, x^n)$, and Lemma 4.11 implies

$$\mathscr{L}\big(C_1^{*(n)}, \dots, C_n^{*(n)}\big) = \mathscr{L}\big(Z_1(x), \dots, Z_{n-1}(x), Z^*_n(x) \,\big|\, T^*_n(x) = n\big)\,.$$

We argue much as in (i), and conclude that

$$\begin{aligned}\frac{1}{a(n)+1} = \frac{p^*(n)}{a^*(n)} &= \frac{p^*(n)x^n(1-x^n)^{p^*(n)}}{\mathbb{P}[T^*_n(x) = n]} \prod_{i=1}^{n-1}(1-x^i)^{p(i)} \\ &= \frac{x^n(1-x^n)}{\mathbb{P}[T^*_n(x) = n]} \prod_{i=1}^{n-1}(1-x^i)^{p(i)}\,.\end{aligned}$$

This yields equation (4.47), and finishes the proof. □

Proof of Theorem 4.33. The AAS $\mathcal{A}$ is quasi-logarithmic. Hence $a(n) > 0$ and $\mathbb{P}\big[T_n(x) = n\big] > 0$ for all n large enough (cf. the discussion at the beginning of Subsection 4.2.1, and Remark 4.30). For any such n we invoke Lemma 4.36, and write

$$a(n) + \delta_n = \frac{\mathbb{P}[T^*_n(x) = n]}{x^n(1-x^n)^{(p(n)\vee 1) - p(n)}} \prod_{i=1}^{n}(1-x^i)^{-p(i)}\,. \tag{4.48}$$

Since $0 < x < 1$, we have

$$\lim_{n\to\infty}(1-x^n)^{(p(n)\vee 1)-p(n)} = 1\,. \tag{4.49}$$

Because $\mathcal{A}$ is (θ, x)-quasi-logarithmic we also have $\operatorname{alim}_{n\to\infty} ip(i)x^i = \theta$ from (4.22). But then Lemma 2.18 implies that $\sup_{i\in\mathbb{N}} ip(i)x^i < \infty$, and so the product $\prod_{i=1}^n \big(\mathrm{e}^{x^i}(1-x^i)\big)^{-p(i)}$ converges to some constant $0 < c' < \infty$ as $n \to \infty$. It follows that

$$\begin{aligned}\prod_{i=1}^{n}(1-x^i)^{-p(i)} &\sim c' \exp\Big(\sum_{i=1}^n p(i)x^i\Big) \\ &= c' \exp\Big(\sum_{i=1}^n \frac{\theta}{i}\Big) \exp\Big(\sum_{i=1}^n \frac{ip(i)x^i - \theta}{i}\Big) \\ &\sim c'\mathrm{e}^{\theta\gamma} n^\theta \ell(n)\,,\end{aligned} \tag{4.50}$$

with $\ell(n)$ as in (4.44). We have $T_n(x) = T^*_n(x)$ if $p(n) > 0$, and

$$\mathbb{P}[T^*_n(x) = n] = \mathbb{P}[T_{n-1}(x) = n](1-x^n) + \mathbb{P}[T_{n-1}(x) = 0]x^n(1-x^n)$$

if $p(n) = 0$. Theorem 3.21 yields

$$\lim_{n\to\infty} n\mathbb{P}\big[T_n(x) = n\big] = p_\theta(1) > 0\,,$$

and therefore

$$\lim_{n\to\infty} n\mathbb{P}\big[T_n^*(x) = n\big] = p_\theta(1) > 0\,, \tag{4.51}$$

since $0 < x < 1$, and so $nx^n \to 0$.

Now (4.48), (4.49), (4.50) and (4.51) lead to

$$a(n) + \delta_n \sim cx^{-n}n^{\theta-1}\ell(n)\,,$$

where $c := c'\mathrm{e}^{\theta\gamma}p_\theta(1) > 0$. Note that (3.17) and (3.20) entail $\mathrm{e}^{\theta\gamma}p_\theta(1) = 1/\Gamma(\theta)$. The asymptotics (4.42) follows from

$$\lim_{n\to\infty} \frac{\delta_n}{x^{-n}n^{\theta-1}\ell(n)} = 0\,.$$

The slow variation of $\ell(n)$ is immediate from $\mathrm{alim}_{n\to\infty}\, ip(i)x^i = \theta$ and Lemma 2.19. □

Corollary 4.37. *Let $\mathcal{A}$ be an (θ, x)-quasi-logarithmic AAS for some $\theta > 0$ and $0 < x < 1$. Then x is the radius of convergence of $\mathcal{A}$, and $\mathbf{A}(x) = \sum_{n=0}^{\infty} a(n)x^n = \infty$.*

Proof. This is immediate from the properties of $\ell(n)$ as a slowly varying function. Indeed, we conclude from (4.42) and (A.7) that

$$\lim_{n\to\infty} \frac{a(n-1)}{a(n)} = x\,,$$

so that x is the radius of convergence of $\mathcal{A}$. Combining (4.27), (4.50) and (A.6) we obtain $\mathbf{A}(x) = \infty$. □

4.3.3.1 Comparison with similar results in the literature

Knopfmacher and Warlimont (2002) examine inverse prime element theorems in an AAS $\mathcal{A}$ that satisfies

$$\lim_{n\to\infty} n^\alpha p(n)x^n = \theta\,, \tag{4.52}$$

for $\theta > 0$, $0 < x < 1$ and $\alpha \in \mathbb{R}$. Their assumption on the component counting function $p(n)$ overlaps with the assumptions made in Theorem 4.33 if $\alpha = 1$;

it then is covered by Corollary 4.34. For $\alpha = 1$, Knopfmacher and Warlimont (2002, Theorem 3.1) obtain

$$\log(x^n a(n)) \sim (\theta - 1)\log n\,, \quad \text{or, equivalently,} \quad a(n) = x^{-n} n^{\theta-1+\mathrm{o}(1)}\,. \tag{4.53}$$

The statement (4.53) involves logarithmic estimates. Knopfmacher and Warlimont (2002, Remark 4.2) write that "*[...] this appears to be unavoidable*". However, Corollary 4.34 shows that the logarithmic estimates can be refined, albeit with an entirely different method of proof.

In order to achieve an asymptotic estimate of $a(n)$ that does not involve logarithms, they refine condition (4.52), with $\alpha = 1$, suitably. These refined assumptions are

$$\sum_{i=1}^{\infty} \Big| \frac{ip(i)x^i - \theta}{i} \Big| < \infty \qquad \text{if } \theta \geq 1,$$

and

$$\sum_{i=n}^{\infty} \Big| \frac{ip(i)x^i - \theta}{i} \Big| = \mathrm{o}\big(n^{\theta-1}\big) \quad \text{or} \quad \sum_{i=1}^{\infty} \sup_{j \geq i} \Big| \frac{jp(j)x^j - \theta}{j} \Big| < \infty \qquad \text{if } 0 < \theta < 1.$$

Knopfmacher and Warlimont (2002, Theorem 4.4) prove that, in our notation,

$$a(n) \sim cLn^{\theta-1}\,, \tag{4.54}$$

with $c > 0$ as in (4.43) and with $L := \lim_{n\to\infty} \ell(n)$, $\ell(n)$ being the function defined in (4.44). The refined assumptions only cover situations where the slowly varying function $\ell(n)$ from Theorem 4.33 is convergent, but may cover examples of AAS that are not quasi-logarithmic.

A simple example of a $(1/2, x)$-quasi-logarithmic AAS that is not covered by the conditions of Knopfmacher and Warlimont is given by $np(n)x^n = 1 + (-1)^{n+1} + \mathrm{o}(1)$ (cf. Example 2.37).

Other versions of an inverse prime element theorem in (θ, x)-logarithmic AAS can be found in Arratia et al. (2003, Subsection 8.5.4) and in Arratia et al. (2005). The proofs of Lemma 4.36 and Theorem 4.33 are adapted from these publications.

In Arratia et al. (2003), it is assumed that, for some $\theta > 0$, $0 < x < 1$ and $\delta > 0$,

$$\big|np(n)x^n - \theta\big| = \mathrm{O}\big(n^{-\delta}\big) \quad \text{and} \quad \big|(n+1)p(n+1)x^{n+1} - np(n)x^n\big| = \mathrm{O}\big(n^{-1-\delta}\big)\,,$$

which leads to an asymptotic behaviour of the counting function $a(n)$ as in (4.54).

In Arratia et al. (2005) we have the condition

$$\sum_{i=1}^{\infty} \frac{1}{i} \sup_{j \geq i} \left| j p(j) x^j - \theta \right| < \infty \,,$$

for $\theta > 0$ and $0 < x < 1$, which leads to (4.54) as well. This condition on $p(n)$ is covered by the refined conditions of Knopfmacher and Warlimont (2002). The proofs, however, are completely different.

5 Applications

5.1 Logical limit laws

A problem that is addressed in finite model theory is to determine the probability that a finite structure, chosen randomly from a class $\mathcal{K}$ of structures, satisfies a given sentence φ of some logic. Here, we examine this problem for classes $\mathcal{K}$ of finite relational structures and a monadic second-order logic, extending previous results of Bell and Burris (2003), Granovsky and Stark (2006) and Stark (2006).

5.1.1 Monadic second-order limit laws

We briefly introduce the notion of a monadic second-order limit law. Burris (2001) serves as main reference. For an introduction to logic and model theory we refer to Ebbinghaus et al. (1984) and Chang and Keisler (1973), respectively.

To construct our monadic second-order logic and to introduce $\mathcal{K}$, we start with a finite purely relational language L, a finite set of relation symbols along with their arities. A finite L-*structure* $\mathbf{S}$ of size n is a pair $(S, \mathcal{J})$, where S is a finite set with n elements, the universe of $\mathbf{S}$, and where $\mathcal{J}$ is an assignment of the relation symbols of L to relations on S which preserves arity. Thus, $\mathbf{S}$ consists of an appropriate interpretation $\mathcal{J}$ of the relation symbols of L in S. Two L-structures $\mathbf{S}_1 = (S_1, \mathcal{J}_1)$ and $\mathbf{S}_2 = (S_2, \mathcal{J}_2)$ are isomorphic if there is a bijection $f : S_1 \to S_2$, such that for each relation R_1 of $\mathbf{S}_1$ and the corresponding relation R_2 of $\mathbf{S}_2$ we have $R_1(x_1, \ldots, x_k)$ if and only if $R_2(f(x_1), \ldots, f(x_k))$ for all $x_1, \ldots, x_k \in S_1$.

A *first-order logic* with language L consists, in addition to the symbols of L, of logical symbols: parentheses, the connectives $\wedge$ (and) and $\neg$ (not), the quantifier $\forall$ (for all), a binary relation symbol $\equiv$ (identity), and first-order variables, which range over elements of structures. A first-order L-formula, that is, roughly speaking, a "meaningful" combination of symbols, is defined inductively using prescribed rules, starting with atomic formulas (cf. Ebbinghaus et al. (1984, Section 3)). A first-order L-sentence is a formula where each variable is bound by a quantifier. For each sentence φ an L-structure will either satisfy or fail to satisfy φ.

A *monadic second-order logic* with language L extends the first-order logic by introducing unary relation variables. These range over subsets of structures. Monadic second-order L-formulas are obtained by augmenting the inductive definition of the first-order formulas by introducing $U(v)$ as an atomic formula for any unary relation variable U and any first-order variable v, and by defining $(\forall U \varphi)$ to be a monadic second-order L-formula if φ is one.

Now fix a class $\mathcal{K}$ of finite L-structures. The notion of probability mentioned at the beginning of this introduction is defined as the limit as $n \to \infty$, if it exists, of the proportion of isomorphy types of structures of size n in $\mathcal{K}$ that satisfy φ among all isomorphy types of structures of size n in $\mathcal{K}$. If this limit exists for all monadic second-order L-sentences, the class $\mathcal{K}$ is said to have a *monadic second-order (local) limit law*.

In the context of a monadic second-order logic with a finite, purely relational language L, Compton (1989) introduced a method of proving such limit laws for $\mathcal{K}$ simply by analyzing the growth rate of $a(n)$, the number of isomorphy types of structures in $\mathcal{K}$ of size n. His method relies on the notion of an adequate class of structures.

Definition 5.1. A class $\mathcal{K}$ of L-structures is *adequate*

(i) if it is closed under disjoint union,

(ii) if elements of $\mathcal{K}$ can be decomposed uniquely, up to commutativity and associativity, into a disjoint union of $\mathcal{K}$-indecomposable structures,

(iii) if the L-structure with empty universe is contained in $\mathcal{K}$,

(iv) and if, up to isomorphism, $\mathcal{K}$ contains only finitely many structures of each size.

If $\mathcal{K}$ is adequate, the set of isomorphism types of structures in $\mathcal{K}$ can be endowed naturally with the structure of an additive arithmetic semigroup $\mathcal{A}_{\mathcal{K}}$. For such a class $\mathcal{K}$, the task of proving the existence of a monadic second-order limit law is reduced to the proof of the existence of asymptotic density of partition sets in $\mathcal{A}_{\mathcal{K}}$. For a comprehensive introduction to Compton's approach to logical limit laws, we refer to Burris (2001).

5.1.2 The density theorem of Compton and Woods

In this subsection, let $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ be an AAS with counting function $a(n)$ of A and $p(n)$ of P, and generating series $\mathbf{A}(z) := \sum_{n=0}^{\infty} a(n) z^n$ of A.

Let $B \subset A$ with counting function $b(n)$. Let $\mathbf{B}(z) := \sum_{n=0}^{\infty} b(n) z^n$ denote the generating series of B. The limit

$$\delta(B) := \lim_{\substack{n\to\infty \\ a(n)\neq 0}} \frac{b(n)}{a(n)}, \tag{5.1}$$

provided it exists, is the *asymptotic density* of B. If the radius of convergence ρ of $\mathcal{A}$ is strictly positive, we define the *Dirichlet density* of $B \subset A$ as the limit

$$\partial(B) := \lim_{x \to \rho^-} \frac{\mathbf{B}(x)}{\mathbf{A}(x)}, \tag{5.2}$$

again provided it exists.

For $m \in \mathbb{Z}_+$ we set

$$mB := \begin{cases} \{e\} & \text{if } m = 0, \\ \{b_1 \circ \cdots \circ b_m : b_j \in B\} & \text{if } m \geq 1, \end{cases}$$

and define

$$(\leq m)B := \bigcup_{j=0}^{m} jB \qquad \text{and} \qquad (\geq m)B := \bigcup_{j=m}^{\infty} jB.$$

Definition 5.2. A set $B \subset A$ is a *partition set* of $\mathcal{A}$ if there is a partition of P into non-empty pairwise disjoint sets $P_1, \ldots, P_k$, and if there are non-negative integers $m_1, \ldots, m_k$, such that

$$B = \mu_1 P_1 \circ \cdots \circ \mu_k P_k,$$

where μ_i is of the form m_i, $(\leq m_i)$ or $(\geq m_i)$ for all $1 \leq i \leq k$.

With these definitions, we can state Compton's results.

Theorem 5.3 (Compton (1989))**.** *Let L be a finite, purely relational language, and let $\mathcal{K}$ be an adequate class of finite L-structures. If all the partition sets of the associated AAS $\mathcal{A}_{\mathcal{K}}$ have asymptotic density, then $\mathcal{K}$ has a local monadic second-order limit law.*

The next result, *Compton's density theorem*, shows that whether or not partition sets have asymptotic density or not depends only on the counting function $a(n)$.

Theorem 5.4 (Compton (1989))**.** *Let $\mathcal{A}$ be a reduced AAS with radius of convergence $0 < \rho < 1$. If there are constants $K > 0$ and $C > 0$, such that*

$$\frac{a(n-k)}{a(n)} \leq C\rho^k \qquad \textit{for all } (k, n) \textit{ with } K \leq k \leq n, \tag{5.3}$$

then all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

Woods's density theorem, a modification of Compton's first result, is used to prove monadic second-order limit laws for classes of unary functions with additional unary predicates.

Theorem 5.5 (Woods (1997))**.** *If there are constants $c > 0$, $0 < x < 1$ and $-\infty < \mu < 1$, such that*

$$p(n) = \mathrm{O}\big(x^{-n}n^{-1}\big) \qquad \textit{and} \qquad a(n) \sim cx^{-n}n^{-\mu}\,,$$

then all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

The first examples of AAS that satisfy the hypotheses of Compton's density theorem were provided by Knopfmacher et al. (1992). Their assumptions on the AAS invoke the counting function $p(n)$ of the set of irreducible elements rather than $a(n)$. These asymptotics were generalized by Bell and Burris (2003).

Theorem 5.6 (Bell and Burris (2003))**.** *Let $\mathcal{A}$ be an AAS with the property that*

$$\lim_{n\to\infty} \frac{p(n-1)}{p(n)} = x \in (0,1) \qquad \textit{and} \qquad \liminf_{n\to\infty} np(n)x^n > 1\,.$$

Then $\mathcal{A}$ is reduced with radius of convergence x, and $a(n)$ satisfies condition (5.3)*.*

Bell and Burris also show that if, for some $0 < x < 1$,

$$p(n) \sim cx^{-n}n^{-\mu}\,, \tag{5.4}$$

where either

$$-\infty < \mu < 1 \qquad \text{and} \qquad c > 0\,, \tag{5.5}$$

or

$$\mu = 1 \qquad \text{and} \qquad c > 1\,, \tag{5.6}$$

then their conditions of Theorem 5.6 are satisfied. Granovsky and Stark (2006) and Stark (2006) have shown that some weakenings of (5.4), cases (5.5) and (5.6), also lead to the conclusions of Theorem 5.6.

Theorem 5.7 (Granovsky and Stark (2006))**.** *Let $\mathcal{A}$ be an AAS such that, for constants $c_1, c_2, r > 0$, $0 < \varepsilon \le r/3$ and $0 < x < 1$,*

$$c_1x^{-n}n^{(2r)/3+\varepsilon-1} \le p(n) \le c_2x^{-n}n^{r-1} \qquad \textit{for all } n \in \mathbb{N}.$$

Then $\mathcal{A}$ is reduced with radius of convergence x, and $a(n)$ satisfies condition (5.3)*.*

Theorem 5.8 (Stark (2006))**.** *Let $\mathcal{A}$ be an AAS such that*

$$1 < \inf_{n\in\mathbb{N}} np(n)x^n \le \sup_{n\in\mathbb{N}} np(n)x^n < \infty\,,$$

where $0 < x < 1$. Then $\mathcal{A}$ is reduced with radius of convergence x, and $a(n)$ satisfies condition (5.3)*.*

5.1.3 Extensions of the Tauberian theorem and density theorem of Woods

We prove an extension of a Tauberian theorem of Woods (1997) and obtain an extension of Theorem 5.5 in this way.

Theorem 5.9. *Let $\mathbf{S}(x) := \sum_{n=0}^{\infty} s(n)x^n$, $\mathbf{T}(x) := \sum_{n=0}^{\infty} t(n)x^n$ be two power series, and let $\mathbf{R}(x) := \sum_{n=0}^{\infty} r(n)x^n$ be the Cauchy product of $\mathbf{S}(x)$ and $\mathbf{T}(x)$, that is*

$$r(n) := \sum_{j+k=n} s(j)t(k) \qquad \textit{for all } n \in \mathbb{Z}_+.$$

If $\mathbf{S}(\rho)$ converges absolutely at ρ for some $0 < \rho \le 1$, and if for some $c > 0$ and $0 \le \mu < 1$

$$s(n) = \mathrm{O}(\rho^{-n}n^{-1})\,, \tag{5.7}$$

$$t(n) \sim c\rho^{-n}n^{-\mu}\ell(n)\,, \tag{5.8}$$

where

$$\ell(n) := \exp\left(\sum_{i=1}^{n} \frac{\theta_i - \theta}{i}\right), \qquad n \in \mathbb{N}\,,$$

and $\{\theta_i\}_{i\in\mathbb{N}}$ is a non-negative $\mathsf{A}(\theta)$-convergent sequence, then

$$\lim_{n\to\infty} \frac{r(n)}{t(n)} = \mathbf{S}(\rho) = \lim_{x\to\rho^-} \frac{\mathbf{R}(x)}{\mathbf{T}(x)}\,. \tag{5.9}$$

Remark 5.10. In contrast to our assumptions, Woods (1997) requires that $t(n) \sim c\rho^{-n}n^{-\mu}$ for some constants $c > 0$ and $-\infty < \mu < 1$.

Proof of Theorem 5.9. We mimic the proof of Woods's Tauberian theorem, as found in Burris (2001, Appendix E).

Recall that $\ell(n)$ is slowly varying because $\{\theta_i\}_{i\in\mathbb{N}}$ is $\mathsf{A}(\theta)$-convergent (cf. Lemma 2.19). From (A.7) we conclude that ρ is the radius of convergence of $\mathbf{T}(x)$.

The second equality in (5.9) is immediate from the fact that $\mathbf{S}(\rho)$ converges absolutely, and that ρ is the radius of convergence of $\mathbf{T}(x)$. In this case $\mathbf{R}(x)$ is equal the usual product $\mathbf{S}(x) \cdot \mathbf{T}(x)$ for all $0 \leq x < \rho$, and $\mathbf{S}(x)$ is continuous on $[0, \rho]$.

To prove the first equality of (5.9), we consider a change of variable $\rho x \mapsto x$. Then we have $\rho = 1$, and we have to show that

$$\lim_{n\to\infty} \left| \mathbf{S}(1) - \frac{r(n)}{t(n)} \right| = 0 \,.$$

We introduce

$$R_n := \sum_{k > \sqrt{n}} \bigl|s(k)\bigr| \qquad \text{and} \qquad M_n := n\sqrt{R_n} + \sqrt{n}\,.$$

Note that $R_n \to 0$, because $\mathbf{S}(1)$ converges absolutely. Also note that $M_n \to \infty$ and $n - M_n > \sqrt{n}$ for all $n \in \mathbb{N}$ large enough. We also have

$$M_n = \mathrm{o}(n) \qquad \text{and} \qquad nR_n = \mathrm{o}(M_n)\,.$$

For all n large enough for $n - M_n > \sqrt{n}$ to hold we consider

$$\begin{aligned}
\left| \mathbf{S}(1) - \frac{r(n)}{t(n)} \right| &\leq \left| \mathbf{S}(1) - \sum_{0 \leq k \leq n} s(k) \right| + \left| \sum_{0 \leq k \leq n} s(k) - \frac{r(n)}{t(n)} \right| \\
&\leq \underbrace{\left| \mathbf{S}(1) - \sum_{0 \leq k \leq n} s(k) \right|}_{U_1(n)} + \underbrace{\sum_{0 \leq k \leq \sqrt{n}} \left| s(k) \Bigl(1 - \frac{t(n-k)}{t(n)} \Bigr) \right|}_{U_2(n)} \\
&\quad + \underbrace{\sum_{\sqrt{n} < k \leq n} \bigl|s(k)\bigr|}_{U_3(n)} + \underbrace{\sum_{\sqrt{n} < k \leq n - M_n} \bigl|s(k)\bigr| \frac{t(n-k)}{t(n)}}_{U_4(n)} \\
&\quad + \underbrace{\sum_{n - M_n < k \leq n} \bigl|s(k)\bigr| \frac{t(n-k)}{t(n)}}_{U_5(n)} \,.
\end{aligned}$$

The first and third expression, $U_1(n)$ and $U_3(n)$, converge to zero, since $\mathbf{S}(1)$ is absolutely convergent by assumption.

To bound $U_2(n)$, let

$$k_n := \operatorname*{arg\,max}_{0 \leq k \leq \sqrt{n}} \left| 1 - \frac{t(n-k)}{t(n)} \right| = \mathrm{o}(n)\,.$$

Invoking (A.8), we obtain from (5.8), with $\rho = 1$, that

$$\frac{t(n-k_n)}{t(n)} \sim \left(\frac{n}{n-k_n}\right)^{\mu} \frac{\ell(n-k_n)}{\ell(n)} \xrightarrow{n\to\infty} 1$$

Invoking the absolute convergence of $\mathbf{S}(1)$ once more, we conclude that

$$U_2(n) \le \left(\sum_{k\ge 0} \left|s(k)\right|\right) \left|1 - \frac{t(n-k_n)}{t(n)}\right| \xrightarrow{n\to\infty} 0\,.$$

To bound $U_4(n)$ from above, a similar argument is used. We introduce

$$k_n' := \operatorname*{arg\,max}_{\sqrt{n}<k\le n-M_n} \frac{t(n-k)}{t(n)}\,,$$

and conclude that

$$U_4(n) \le R_n \frac{t(n-k_n')}{t(n)}\,. \tag{5.10}$$

Since $n - k_n' \ge M_n \to \infty$, we obtain from (5.8) with $\rho = 1$ that

$$\frac{t(n-k_n')}{t(n)} \sim \left(\frac{n}{n-k_n'}\right)^{\mu} \frac{\ell(n-k_n')}{\ell(n)}\,. \tag{5.11}$$

From the representation theorem for slowly varying sequences, Theorem A.3, it follows that

$$\frac{\ell(n-k_n')}{\ell(n)} \sim \exp\left(-\sum_{i=n-k_n'+1}^{n} \frac{\delta_i}{i}\right) \le \left(\frac{n}{n-k_n'}\right)^{\varepsilon_n}, \tag{5.12}$$

where $\delta_i \to 0$ and $\varepsilon_n := \sup_{i\ge M_n} |\delta_i|$. Using inequalities (5.11) and (5.12) together with (5.10) we conclude that, for some constant $c_1 > 0$,

$$U_4(n) \le c_1 R_n \left(\frac{n}{n-k_n'}\right)^{\mu+\varepsilon_n} \le c_1 R_n \left(\frac{n}{M_n}\right)^{\mu+\varepsilon_n}.$$

By assumption we have $0 \le \mu < 1$. Because $\varepsilon_n \to 0$ by construction, we have $\mu \le \mu + \varepsilon_n < 1$ for all $n \in \mathbb{N}$ large enough. For these n we have

$$U_4(n) \le c_1 R_n^{1-\mu-\varepsilon_n} \left(\frac{nR_n}{M_n}\right)^{\mu+\varepsilon_n}. \tag{5.13}$$

Recall that $R_n \to 0$ and $nR_n = \mathrm{o}(M_n)$. Therefore, the right hand side of (5.13), and thus $U_4(n)$, converges to zero for every $0 \le \mu < 1$.

To bound $U_5(n)$, we rewrite this expression as

$$U_5(n) = \sum_{0\le k<M_n} \big|s(n-k)\big| \frac{t(k)}{t(n)} \le \big|s(n-k_n'')\big| \frac{t(0)}{t(n)} + \big|s(n-k_n'')\big| \sum_{1\le k<M_n} \frac{t(k)}{t(n)}\,, \tag{5.14}$$

where

$$k_n'' := \operatorname*{arg\,max}_{0\le k<M_n} \big|s(n-k)\big| = \mathrm{o}(n)\,.$$

Under assumption (5.7), with $\rho = 1$, we have $\big|s(n-k_n'')\big| \sim 1/n$, so that, since $\ell(n)$ is slowly varying and therefore $n^{1-\mu}\ell(n) \to \infty$ for all $0 \le \mu < 1$ (cf. (A.6)),

$$\big|s(n-k_n'')\big| \frac{t(0)}{t(n)} \sim \frac{t(0)}{n^{1-\mu}\ell(n)} \xrightarrow{n\to\infty} 0\,. \tag{5.15}$$

Furthermore, it follows from (5.8) that, for some constant $c_2 > 0$,

$$\big|s(n-k_n'')\big| \sum_{1\le k<M_n} \frac{t(k)}{t(n)} \le c_2 \frac{1}{n} \sum_{1\le k<M_n} \Big(\frac{n}{k}\Big)^{\mu} \frac{\ell(k)}{\ell(n)}\,. \tag{5.16}$$

By assumption, $\{\theta_i\}_{i\in\mathbb{N}}$ is A-convergent, that is

$$\tilde\theta_n(m_n,\theta) := \max_{0\le j\le \lfloor n/m_n\rfloor} \left| \frac{1}{m_n} \sum_{i=1}^{m} \theta_{jm_n+i} - \theta \right|$$

converges to 0 as $n \to \infty$, for every sequence $\{m_n\}_{n\in\mathbb{N}}$ of positive integers with $m_n \to \infty$ and $m_n = \mathrm{o}(n)$. Recall that Lemma 2.18 yields $\theta_{\sup} := \sup_{i\in\mathbb{N}} \theta_i < \infty$. Here, we choose the sequence $\{m_n\}_{n\in\mathbb{N}}$ defined by

$$m_n := \left\lfloor \Big(\frac{n}{M_n}\Big)^{(1-\mu)/(2(\theta_{\sup}\vee 1))} \right\rfloor \vee 1 \qquad \text{for all } n \in \mathbb{N}.$$

Since $M_n = \mathrm{o}(n)$ and $0 < (1-\mu)/(2(\theta_{\sup}\vee 1)) < 1$ we have $m_n \to \infty$ and $m_n = \mathrm{o}(n)$ as required. Now invoke Lemma A.16 (ii) (with $x_i := \theta_i$, $x := \theta$, $l := k$, $m := m_n$, and noting that $\theta_{\sup} \ge \theta$ under $\mathsf{A}(\theta)$-convergence). We obtain for all $n \in \mathbb{N}$, such that $M_n \le n$,

$$\begin{aligned} \frac{\ell(k)}{\ell(n)} &\le \chi^{\theta_{\sup}} \Big(\frac{n}{M_n}\Big)^{\theta_{\sup}(1-\mu)/(2(\theta_{\sup}\vee 1))} \Big(\frac{n}{k}\Big)^{\tilde\theta_n(m_n,\theta)} \\ &\le \chi^{\theta_{\sup}} \Big(\frac{n}{M_n}\Big)^{(1-\mu)/2} \Big(\frac{n}{k}\Big)^{\tilde\theta_n(m_n,\theta)} \qquad \text{for all } 1 \le k < M_n, \end{aligned}$$

where $\chi := \exp(1 + \pi^2/6 + \log 2)$. Combining this result with (5.16) implies that

$$
\begin{aligned}
&\big|s(n-k_n'')\big| \sum_{1\le k<M_n} \frac{t(k)}{t(n)} \\
&\qquad\qquad \le c_2 \chi^{\theta_{\sup}} \frac{1}{n^{1-\mu-\tilde\theta_n(m_n,\theta)}} \Big(\frac{n}{M_n}\Big)^{(1-\mu)/2} \sum_{1\le k<M_n} \frac{1}{k^{\mu+\tilde\theta_n(m_n,\theta)}}
\end{aligned}
$$

Because $0 \le \mu < 1$ and $\tilde\theta_n(m_n,\theta) \to 0$ we have $\mu \le \mu + \tilde\theta_n(m_n,\theta) < 1$ for all $n \in \mathbb{N}$ large enough, and for these n

$$
\sum_{1\le k<M_n} \frac{1}{k^{\mu+\tilde\theta_n(m_n,\theta)}} \le \frac{M_n^{1-\mu-\tilde\theta_n(m_n,\theta)}}{1-\mu-\tilde\theta_n(m_n,\theta)}
$$

holds true. Hence, we conclude that

$$
\big|s(n-k_n'')\big| \sum_{1\le k<M_n} \frac{t(k)}{t(n)} \le \frac{c_2 \chi^{\theta_{\sup}}}{1-\mu-\tilde\theta_n(m_n,\theta)} \Big(\frac{M_n}{n}\Big)^{(1-\mu)/2-\tilde\theta_n(m_n,\theta)} . \tag{5.17}
$$

The right hand side of (5.17) converges to zero because of $M_n = \mathrm{o}(n)$ and $\tilde\theta_n(m_n,\theta) \to 0$. Finally, combining (5.14) with (5.15) and (5.17) yields $U_5(n) \to 0$, which proves the theorem. □

Theorem 5.11. *Let $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ be an AAS with counting functions $a(n)$ of A and $p(n)$ of P. Assume that, for some $0 < \rho \le 1$, $c > 0$ and $0 \le \mu < 1$,*

$$
\begin{aligned}
p(n) &= \mathrm{O}(\rho^{-n} n^{-1})\,, && (5.18)\\
a(n) &\sim c\rho^{-n} n^{-\mu} \ell(n)\,, && (5.19)
\end{aligned}
$$

where

$$
\ell(n) := \exp\Big(\sum_{i=1}^{n} \frac{\theta_i - \theta}{i}\Big), \qquad n \in \mathbb{N},
$$

and $\{\theta_i\}_{i\in\mathbb{N}}$ is a non-negative $\mathsf{A}(\theta)$-convergent sequence. Then all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

Proof. It follows from $a(n) \sim c\rho^{-n} n^{-\mu} \ell(n)$ and the slow variation of $\ell(n)$ (cf. Lemma 2.19 and (A.8)) that ρ is the radius of convergence of $\mathcal{A}$. The remainder of the proof is exactly the same as the proof of Woods's density theorem in Burris (2001, Appendix E), so we only give a short outline. Recall the definition of the asymptotic density δ and Dirichlet density ∂ in (5.1) and (5.2), respectively.

If $\rho = 1$, then Burris (2001, Theorem 4.2) states that $\lim_{n\to\infty} a(n-1)/a(n) = 1$ is sufficient for all partition sets to have asymptotic density, which agrees with the Dirichlet density on these sets.

Now let $0 < \rho < 1$, and let B be a partition set. Note that the Dirichlet density $\partial(b)$ exists by Burris (2001, Theorem 3.40). If $\partial(B) = 0$, then the asymptotic density $\delta(B)$ is zero as well (cf. Burris (2001, Corollary 5.7)).

It remains to prove the theorem in the case where $\partial(B) > 0$. Here, Lemma 5.9 is applied with special forms of the power series $\mathbf{S}(x) = \sum_{n=0}^{\infty} s(n)x^n$ and $\mathbf{T}(x) = \sum_{n=0}^{\infty} t(n)x^n$, namely

$$\mathbf{S}(x) := \bar{\mathbf{B}} * [1/\mathbf{A}](x) \qquad \text{and} \qquad \mathbf{T}(x) := \mathbf{A}(x)\,,$$

where $\mathbf{A}(x)$ is the generating series of the AAS $\mathcal{A}$, $[1/\mathbf{A}](x)$ is the power series expansion of $1/\mathbf{A}(x)$, $\bar{\mathbf{B}}(x) := \sum_{n=0}^{\infty} \bar{b}(n)x^n$ is the generating series of a special partition set $\bar{B}$, constructed from B (see Burris (2001, p. 92) for details), and where $*$ denotes the Cauchy product. In this special case, the conclusion (5.9) of Theorem 5.9 translates into

$$\lim_{n\to\infty} \frac{\bar{b}(n)}{a(n)} = \mathbf{S}(\rho) = \lim_{x\to\rho} \frac{\bar{\mathbf{B}}(x)}{\mathbf{A}(x)}\,,$$

which means that $\delta(\bar{B}) = \partial(\bar{B})$. But $\delta(\bar{B}) = \delta(B)$ and $\partial(\bar{B}) = \partial(B)$ (Burris, 2001, Lemma 5.12).

The assumptions of Theorem 5.9 hold true with our choices of $\mathbf{S}(x)$ and $\mathbf{T}(x)$. Indeed, since $t(n) = a(n)$ for all $n \in \mathbb{N}$, (5.19) immediately yields (5.8), and Burris (2001, p. 278–279) proves that (5.18) implies (5.7). What is more, it follows from Burris (2001, Corollary 5.11 and p. 10 above) that $\mathbf{S}(\rho)$ converges absolutely. $\square$

5.1.4 Density in quasi-logarithmic AAS and logical limit laws

Collecting the results from the previous subsections, we now prove the main theorems of Section 5.1.

Theorem 5.12. *Let $\theta > 0$ and let $0 < x < 1$. Let $\mathcal{A}$ be an (x, θ)-quasi-logarithmic AAS. Then all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.*

Proof. We treat the cases $0 < \theta \le 1$ and $\theta > 1$ separately. In the first case we apply our extension of Woods' s density theorem, Theorem 5.11, whereas for the second case we invoke Compton's density theorem, Theorem 5.4.

First, assume that $0 < \theta \le 1$. Lemma 2.18 implies (5.18) with $\rho := x$, and Theorem 4.33 yields (5.19) with $\rho := x$, $\mu := 1 - \theta$ and $\theta_i := ip(i)x^i$, $i \in \mathbb{N}$, where $p(i)$ is the component counting function of $\mathcal{A}$. Hence, the assumptions of Theorem 5.11 are valid, and we infer that all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

Now, let $\theta > 1$. Corollary 4.37 implies that x is the radius of convergence of $\mathcal{A}$. We show that there are constants $C, K > 0$, such that (5.3) holds true with $\rho := x$. Then Theorem 5.4 implies that all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

Let $k = n$. It follows from (4.42) and because $a(0) = 1$, that

$$\frac{a(0)}{a(n)} \sim \frac{1}{cn^{\theta-1}\ell(n)} x^n .$$

Because $\ell(n)$ is slowly varying (cf. Lemma 2.19) and because $\theta > 1$, (A.6) yields $n^{\theta-1}\ell(n) \to \infty$. Since $\mathcal{A}$ is reduced (cf. Remark 4.30), there is a $K_0 > 0$, such that $a(n) \ge 1$ for all $n \ge K_0$. It follows that, for some $c_1 > 0$,

$$\frac{a(0)}{a(n)} \le c_1 x^n \qquad \text{for all } n \ge K_0.$$

If $0 \le k < n$, (4.42) implies that, for some $c_2 > 0$,

$$\frac{a(n-k)}{a(n)} \le c_2 \Big(\frac{n-k}{n}\Big)^{\theta-1} \frac{\ell(n-k)}{\ell(n)} x^k \qquad \text{for all } n \ge K_0. \tag{5.20}$$

By assumption, the sequence $\theta_i := ip(i)x^i$, $i \in \mathbb{N}$, is $\mathsf{A}(\theta)$-convergent; that is,

$$\tilde{\theta}_n(m_n, \theta) := \max_{0 \le j \le \lfloor n/m_n \rfloor} \left| \frac{1}{m_n} \sum_{i=1}^{m} \theta_{jm_n+i} - \theta \right|$$

converges to 0 as $n \to \infty$, for every sequence $\{m_n\}_{n\in\mathbb{N}}$ of positive integers with $m_n \to \infty$ and $m_n = \mathrm{o}(n)$. Note that $\theta_{\sup} := \sup_{i\in\mathbb{N}} \theta_i < \infty$ from Lemma 2.18. Let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence such that there is a $K_1 > K_0$ with

$$m_n \le n^{(\theta-1)/(\theta-1+2\theta_{\sup})} \qquad \text{for all } n \ge K_1. \tag{5.21}$$

Then we choose $K > K_1$, such that

$$\theta - 1 - \tilde{\theta}_n(m_n, \theta) \ge \frac{\theta - 1}{2} \qquad \text{for all } n \ge K. \tag{5.22}$$

Let $n \geq K$. If $n - m_n < k < n$, we conclude from (5.20), using Lemma A.16 (ii) (with $x_i := \theta_i$, $x := \theta$, $l := n - k$, $m := m_n$), (5.22) and (5.21), that

$$\begin{aligned}\frac{a(n-k)}{a(n)} &\leq c_2 \chi_1^{\theta_{\sup}} m_n^{\theta_{\sup}} \left(\frac{n-k}{n}\right)^{(\theta-1)/2} x^k \\ &\leq c_2 \chi_1^{\theta_{\sup}} m_n^{\theta_{\sup}} \left(\frac{m_n}{n}\right)^{(\theta-1)/2} x^k \\ &\leq c_2 \chi_1^{\theta_{\sup}} x^k ,\end{aligned}$$

where $\chi_1 := \exp(1 + \pi^2/6 + \log 2)$.

If $0 \leq k \leq n - m_n$, we obtain from (5.20), using Lemma A.16 (iii) (again with $x_i := \theta_i$, $x := \theta$, $l := n - k$, $m := m_n$) and (5.22), that

$$\frac{a(n-k)}{a(n)} \leq c_2 \chi_2^{\theta_{\sup}} \left(\frac{n-k}{n}\right)^{(\theta-1)/2} x^k \leq c_2 \chi_2^{\theta_{\sup}} x^k ,$$

with $\chi_2 := \exp(\pi^2/6 + 2 \log 2)$.

Now (5.3) follows with K as above, $C := \max\{c_1, c_2\chi_1^{\theta_{\sup}}, c_2\chi_2^{\theta_{\sup}}\}$ and $\rho := x$. This proves the theorem. □

Remark 5.13. If $\mathcal{A}$ is a (θ, x)-quasi-logarithmic AAS with $0 < \theta < 1$, then Compton's density theorem (Theorem 5.4) cannot be applied because assumption (5.3) is violated.

To see this, assume that there are constants $C > 0$ and $K > 0$ such that (5.3) holds. Recalling from Corollary 4.37 that the radius of convergence in a (θ, x)-quasi-logarithmic AAS is x, and choosing $k := n-1$, (5.3) translates into

$$\frac{a(1)}{a(n)} \leq C x^{n-1} \qquad \text{for all } n \text{ with } n > K. \tag{5.23}$$

But it follows from Theorem 4.33 that there is a constant $c' > 0$, such that

$$\frac{a(1)}{a(n)} = \frac{1}{a(n)} \geq c' \frac{1}{n^{\theta-1}\ell(n)} x^{n-1} \qquad \text{for all } n \in \mathbb{N}. \tag{5.24}$$

Moreover, $\lim_{n\to\infty} n^{\theta-1}\ell(n) = 0$ by (A.6), because $\theta < 1$. Thus, (5.24) contradicts (5.23).

As a consequence of Theorem 5.12 and Theorem 5.3 we obtain the following logical limit law.

Theorem 5.14. *Let* L *be a finite, purely relational language, and let* $\mathcal{K}$ *be an adequate class of finite* L*-structures, such that the associated AAS* $\mathcal{A}_{\mathcal{K}}$ *is quasi-logarithmic. Then* $\mathcal{K}$ *has a monadic second-order limit law.*

5.1.4.1 Comparison with density theorems in the literature

Let $\theta > 0$ and $0 < x < 1$. Let $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ be a (θ, x)-quasi-logarithmic AAS with counting functions $a(n)$ of A and $p(n)$ of P. Theorem 5.12 covers various situations which which are not covered by the results obtained so far (cf. the theorems in Subsection 5.1.2 and the references therein).

(i) If $0 < \theta < 1$ then Remark 5.13 entails that the assumption of Compton's density theorem, Theorem 5.4, does not hold. Therefore, Theorems 5.6, 5.7 and 5.8 cannot be applied either. Note that this class of AAS particularly includes all (θ, x)-logarithmic AAS, which satisfy

$$\lim_{n\to\infty} np(n)x^n = \theta \in (0,1)\,.$$

(ii) Assume that the slowly varying function $\ell(n)$, defined in (4.44), does not converge as $n \to \infty$. Then Theorem 4.33 shows that $a(n)$ has an asymptotic behaviour that is not covered by Woods's density theorem, Theorem 5.5. Recall from Remark 4.35 that even if $\ell(n)$ arises from a *logarithmic* AAS it can exhibit any asymptotic growth behaviour that a slowly varying function can have.

(iii) If $\theta > 1$ and $\liminf_{n\to\infty} np(n)x^n \le 1$, then the requirements of Compton's density theorem are true, as follows from the proof of Theorem 5.12, but $p(n)$ does not satisfy the conditions of Theorems 5.6, 5.7 or 5.8.

Example 5.15. A simple example of an AAS $\mathcal{A}$ that is neither covered by Compton's nor by Woods's density theorem is given by

$$p(n) = \big(1/2 + (\log n)^{-1}\big)x^{-n}n^{-1} + \mathrm{O}(1) \qquad \text{for all } n \in \mathbb{N},\ n \ge 2.$$

Then $\mathcal{A}$ is $(1/2, x)$-logarithmic. By Theorem 5.12, all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density. It follows from Remark 5.13 that $\mathcal{A}$ does not satisfy the assumptions of Theorem 5.4. What is more, we have for some $K > 0$

$$\begin{aligned}\left|\sum_{i=1}^{n} \frac{ip(i)x^i - 1/2}{i} - \log\log n\right| &\le \left|\sum_{i=1}^{n} \frac{ip(i)x^i - 1/2}{i} - \sum_{i=2}^{n} \frac{1}{i\log i}\right| \\ &\quad + \left|\sum_{i=2}^{n} \frac{1}{i\log i} - \log\log n\right| \\ &\le K\,.\end{aligned}$$

This yields

$$\ell(n) := \exp\left(\sum_{i=1}^{n} \frac{ip(i)x^i - 1/2}{i}\right) \asymp \log n\,,$$

so that (4.42) entails

$$a(n) \asymp x^{-n} n^{-1/2} \log n \, .$$

Thus, $\mathcal{A}$ does not satisfy the requirements of Theorem 5.5 either.

5.2 Distributional limits of quasi-logarithmic structures

In this section we return to general quasi-logarithmic random decomposable combinatorial structures as introduced in Definition 4.17. For this, let $\mathfrak{Z} := \{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables. As in Subsection 2.1.1 we assume that there is a positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ and a family $\{Z_{ij} : i \in \mathbb{N} \text{ and } 1 \le j \le r_i\}$ of independent $\mathbb{Z}_+$-valued random variables, such that $Z_{i1}, \ldots, Z_{ir_i}$ are identically distributed for each $i \in \mathbb{N}$, and such that $Z_i = \sum_{j=1}^{r_i} Z_{ij}$ for all $i \in \mathbb{N}$. Let $\theta_i := i\mathbb{E}Z_i$ for all $i \in \mathbb{N}$, and $\theta_{\sup} := \sup_{i\in\mathbb{N}} \theta_i$. Recall the definition of $\varepsilon_{ik}(\theta_i, r_i)$ in (2.3) and the definition of $\mu_i(\mathfrak{r})$ in (2.6). We also set

$$T_{k,l} := \sum_{i=k+1}^{l} iZ_i \qquad \text{for all } k \in \mathbb{Z}_+ \text{ and } l \in \mathbb{N} \text{ with } k < l, \tag{5.25}$$

to save on notation, and finally let $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ be an RDCS.

5.2.1 The size of the largest component

The following theorem is an extension of a result proved by Kingman (1977) in the case of θ-tilted random permutations. Our proof is an adaptation of a corresponding result for logarithmic RDCS found in Arratia et al. (2003, Lemma 5.7)

Theorem 5.16. *Let $\theta > 0$, and let $\mathfrak{Z}$ satisfy condition* $\mathsf{SUQLC}(\theta, \mathfrak{r})$. *Assume that $\mathfrak{C}$ is $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic. Let*

$$L^{(n)} := \max\bigl\{1 \le i \le n : C_i^{(n)} > 0\bigr\}$$

the length of the largest component. Then

$$\lim_{n\to\infty} \mathscr{L}\bigl(n^{-1} L^{(n)}\bigr) = \mathscr{L}(L) \, ,$$

where L is a random variable concentrated on $(0, 1]$, whose distribution is given by the density function

$$f_\theta(x) := \mathrm{e}^{\gamma\theta}\Gamma(\theta+1)x^{\theta-2} p_\theta\bigl((1-x)/x\bigr) \, , \qquad \textit{for } 0 < x \le 1.$$

In particular, if $\theta = 1$,

$$\lim_{n\to\infty} \mathbb{P}\big[L^{(n)} \le n/y\big] = \rho_1(y)\,, \qquad \text{for all } y \ge 1,$$

where ρ_1 is Dickman's function (cf. Definition 3.11).

Proof. Fix $x \in (0,1]$. It follows that

$$\begin{aligned}\mathbb{P}\big[n^{-1}L^{(n)} \le x\big] &= \mathbb{P}\big[L^{(n)} \le \lfloor nx\rfloor\big] \\ &= \mathbb{P}\big[C^{(n)}_{\lfloor nx\rfloor+1} = \cdots = C^{(n)}_n = 0\big] \\ &= \prod_{i=\lfloor nx\rfloor+1}^{n} \mathbb{P}\big[Z_i = 0\big]\,\frac{\mathbb{P}\big[T_{0,\lfloor nx\rfloor} = n\big]}{\mathbb{P}\big[T_{0,n} = n\big]}\,.\end{aligned}$$

Theorem 3.21 implies

$$\lim_{n\to\infty} \frac{\mathbb{P}\big[T_{0,\lfloor nx\rfloor} = n\big]}{\mathbb{P}\big[T_{0,n} = n\big]} = \frac{p_\theta(1/x)}{x p_\theta(1)}\,.$$

The sequence $\mathfrak{Z} = \{Z_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{SUQLC}(\theta,\mathfrak{r})$, by assumption. Recall from the beginning of Subsection 2.1.1 that

$$\prod_{i=\lfloor xn\rfloor+1}^{n} \mathbb{P}\big[Z_i = 0\big] = \prod_{i=\lfloor xn\rfloor+1}^{n}\prod_{j=1}^{r_i} \mathbb{P}\big[Z_{ij} = 0\big] = \prod_{i=\lfloor xn\rfloor+1}^{n} \mathbb{P}\big[Z_{i1} = 0\big]^{r_i}\,, \quad (5.26)$$

where

$$\mathbb{P}\big[Z_{i1} = 0\big] = 1 - \frac{\mathbb{E}Z_i}{r_i}\left(1 + \sum_{k=1}^{\infty} \varepsilon_{ik}(\theta_i, r_i)\right) \qquad \text{for all } i \in \mathbb{N}.$$

We write

$$E_i := \sum_{k=1}^{\infty} \varepsilon_{ik}(\theta_i, r_i) \qquad \text{and} \qquad s_i := \frac{\theta_i}{i r_i}(1 + E_i)$$

to save on notation. Equation (5.26) translates into

$$\prod_{i=\lfloor xn\rfloor+1}^{n} \mathbb{P}\big[Z_i = 0\big] = \exp\left(-\sum_{i=\lfloor xn\rfloor+1}^{n} \mathbb{E}Z_i(1 + E_i)\right) \prod_{i=\lfloor xn\rfloor+1}^{n} \left(\frac{1 - s_i}{\mathrm{e}^{-s_i}}\right)^{r_i}.$$

To show that

$$\lim_{n\to\infty} \prod_{i=\lfloor xn\rfloor+1}^{n} \left(\frac{1 - s_i}{\mathrm{e}^{-s_i}}\right)^{r_i} = 1\,, \qquad (5.27)$$

we recall from (A.14) that $-1 \leq E_i \leq 0$ for all $i \in \mathbb{N}$. Also, recall from Lemma 2.18 that $\mathbb{E}Z_i = \mathrm{O}(1/i)$ (because $\operatorname{alim}_{i\to\infty} \theta_i = \theta$ under condition $\mathsf{SUQLC}(\theta, \mathfrak{r})$). It follow that there is a $c > 0$ such that

$$0 \leq s_i \leq \frac{c}{ir_i} \qquad \text{for all } i \in \mathbb{N}.$$

Now choose an $n_0 \in \mathbb{N}$ such that $c/(ir_i) < 1$ for all $i \geq n_0$. For any such i we conclude that

$$\begin{aligned}
\sum_{i=\lfloor xn \rfloor+1}^{n} r_i \bigl|\log(1-s_i) + s_i\bigr| &= \sum_{i=\lfloor xn \rfloor+1}^{n} r_i \sum_{k=2}^{\infty} \frac{s_i^k}{k} \\
&\leq \sum_{i=\lfloor xn \rfloor+1}^{n} r_i s_i^2 \sum_{k=0}^{\infty} s_i^k \\
&\leq \sum_{i=\lfloor xn \rfloor+1}^{n} \frac{c^2}{r_i i^2} \frac{1}{1-s_i}\,.
\end{aligned}$$

The right hand side converges to 0 as $n \to \infty$. This entails (5.27).

To prove that

$$\lim_{n\to\infty} \exp\Bigl(-\sum_{i=\lfloor xn \rfloor+1}^{n} \mathbb{E}Z_i(1+E_i)\Bigr) = x^{\theta}\,,$$

we split this exponential function into the product

$$\exp\Bigl(-\sum_{i=\lfloor nx \rfloor+1}^{n} \frac{\theta}{i}\Bigr) \exp\Bigl(-\sum_{i=\lfloor nx \rfloor+1}^{n} \frac{\theta_i - \theta}{i}\Bigr) \exp\Bigl(-\sum_{i=\lfloor nx \rfloor+1}^{n} \mathbb{E}Z_i E_i\Bigr)\,.$$

The first factor converges to x^{θ}; the second to 1 (cf. Lemma 2.19). The third factor converges to 1 as well, because, if $\lfloor nx \rfloor \geq 1$,

$$\Bigl|\sum_{i=\lfloor nx \rfloor+1}^{n} \mathbb{E}Z_i E_i\Bigr| \leq \sum_{i=\lfloor nx \rfloor+1}^{n} \mathbb{E}Z_i |E_i| \leq \sup_{i\geq\lfloor nx \rfloor} \mu_i(\mathfrak{r}) \sum_{i=\lfloor nx \rfloor+1}^{n} \mathbb{E}Z_i\,,$$

with $\mu_i(\mathfrak{r})$ defined in (2.6). The right hand side converges to 0, because under our quasi-logarithmic condition we have $\mu_i(\mathfrak{r}) \to 0$ and $\mathbb{E}Z_i = \mathrm{O}(1/i)$.

Thus, in all, we have

$$\lim_{n\to\infty} \mathbb{P}\bigl[n^{-1}L^{(n)} \leq x\bigr] = \frac{x^{\theta} p_{\theta}(1/x)}{x p_{\theta}(1)} =: F_{\theta}(x)\,,$$

where F_{θ} is a distribution function with density f_{θ} (Arratia et al., 2003, pp. 108/109). Recall from (3.20) that $p_1(x) = e^{-\gamma}\rho_1(x)$. The theorem follows. $\square$

5.2.2 An analogue of the Kubilius fundamental lemma for quasi-logarithmic RDCS

We start with a simple theorem on the asymptotic behaviour of the spectrum of small components.

Theorem 5.17. *Let $\theta > 0$, and let $\mathfrak{Z}$ satisfy condition* $\mathsf{SUQLC}(\theta, \mathfrak{r})$*. If $\mathfrak{C}$ is $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic, then we have*

$$\lim_{n\to\infty} \mathscr{L}_{\nu_n}\big(C_1^{(n)}, \dots C_a^{(n)}\big) = \mathscr{L}\big(Z_1, \dots, Z_a\big)$$

for every fixed $a \in \mathbb{N}$.

Proof. Fix $a \in \mathbb{N}$. Let $(c_1, \dots, c_a) \in \mathbb{Z}_+^a$. Because $\mathfrak{C}$ is quasi-logarithmic, there is an $n_0 > a$ such that for all $n \geq n_0$, $\Omega_n \neq \emptyset$, $\mathbb{P}[T_{0,n} = n] > 0$ and

$$\begin{aligned} &\mathbb{P}_{\nu_n}\big[(C_1^{(n)}, \dots, C_a^{(n)}) = (c_1, \dots, c_a)\big] \\ &\quad = \mathbb{P}\big[(Z_1, \dots, Z_a) = (c_1, \dots, c_a) \;\big|\; T_{0,n} = n\big] \\ &\quad = \mathbb{P}\big[(Z_1, \dots, Z_a) = (c_1, \dots, c_a)\big] \, \frac{\mathbb{P}\big[T_{a,n} = n - \sum_{i=1}^a c_i\big]}{\mathbb{P}\big[T_{0,n} = n\big]} \,. \end{aligned}$$

Theorem 3.21 implies that

$$\lim_{n\to\infty} \frac{\mathbb{P}\big[T_{a,n} = n - \sum_{i=1}^l c_i\big]}{\mathbb{P}\big[T_{0,n} = n\big]} = 1 \,,$$

which completes the proof. □

In order to obtain total variation approximations sharpening Theorem 5.17, we additionally assume that

$$\sum_{j=1}^{\infty} \frac{|\varepsilon_{j1}(\theta_j, r_j)|}{j} < \infty \,, \tag{5.28}$$

recalling that $\theta_j := j\mathbb{E}Z_j$, and the definition of $\varepsilon_{j1}(\theta_j, r_j)$ from (2.3). Note that (5.28) is a very mild assumption. It is satisfied if

$$\sum_{i=1}^{\infty} \frac{\mu_i(\mathfrak{r})}{i} < \infty \,,$$

with $\mu_i(\mathfrak{r})$ given in (2.6), a condition which in turn is satisfied if the Z_i have distributions such as the Poisson, Binomial or negative Binomial distributions from Subsection 2.1.2.

Recalling that $\theta_i := i\mathbb{E}Z_i$, we also define $\theta_{\sup} := \sup_{i\in\mathbb{N}} \theta_i$. Note that $\theta_{\sup} < \infty$, because non-negative A-convergent sequences are bounded (cf. Lemma 2.18). Moreover, let

$$\tilde{\theta}_n(m,\theta) := \max_{0\le j\le \lfloor n/m\rfloor} \left| \frac{1}{m}\sum_{i=1}^{m} \theta_{jm+i} - \theta \right| \qquad \text{for all } m \in \mathbb{N}.$$

For $k \in \mathbb{Z}_+$ and $l \in \mathbb{N}$ with $k < l$, recall the definition of $T_{k,l}$ in (5.25), and define

$$\Delta_{k,l}(j) := \left| j\mathbb{P}[T_{k,l} = j] - \sum\nolimits_{i=k+1}^{l} \theta_i \mathbb{P}[T_{k,l} = j-i] \right| \qquad \text{for all } j \in \mathbb{N}, \quad (5.29)$$

and

$$\Delta^*_{k,l} := \sup\nolimits_{j\in\mathbb{N}} \Delta_{k,l}(j)\,.$$

It follows that

$$\mathbb{P}[T_{k,l} = j] \le \frac{\theta_{\sup} + \Delta^*_{k,l}}{j} \qquad \text{for all } j \in \mathbb{N}, \tag{5.30}$$

and, in combination with the Markov inequality,

$$\mathbb{P}[T_{k,l} = j] \le \frac{\theta_{\sup}\mathbb{E}T_{k,l}}{j(j-l)} + \frac{\Delta^*_{k,l}}{j} \le \frac{\theta^2_{\sup} l}{j(j-l)} + \frac{\Delta^*_{k,l}}{j} \qquad \text{for all } j > l. \tag{5.31}$$

Remark 5.18. If $T_{k,l}$ has a compound Poisson distribution with rates $\lambda_i := \theta_i/i$ for $k < i \le l$ and $\lambda_i := 0$ else, it follows from Corollary 3.3 that $\Delta^*_{k,l} = 0$.

Theorem 5.19. *Let $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ be an RDCS which satisfies the conditioning relation (4.8) with the sequence $\mathfrak{Z}$. Assume that (5.28) and that $\theta_{\sup} < \infty$ hold. Let $\theta > 0$. Let $a, m \in \mathbb{N}$, and let $n \ge \max\{4a, 2(a+m)\}$ be such that $\Omega_n \ne \emptyset$ and $\mathbb{P}[T_{0,n} = n] > 0$. Then we have*

$$\begin{aligned}
&d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C_1^{(n)}, \dots, C_a^{(n)}), \mathscr{L}(Z_1, \dots, Z_a)\big) \\
&\quad\le \frac{c\lambda^2_{a,n}}{(n\mathbb{P}[T_{0,n} = n]) \wedge 1} \Bigg(\frac{a}{n} + m^2 d_{\mathrm{TV}}\big(\mathscr{L}(T_{a,n}), \mathscr{L}(T_{a,n}+1)\big) + \tilde{\theta}_n(m,\theta) \\
&\qquad + \Big(\frac{m}{n}\Big)^{(\theta - \tilde{\theta}_n(m,\theta))\wedge 1} + \Delta^*_{a,n} + \min\big\{\Delta^*_{0,a},\, \sup\nolimits_{j\in\mathbb{Z}_+} \mathbb{P}[T_{a,n} = j]a\big\}\Bigg) \wedge 1\,,
\end{aligned}$$

with $\lambda_{a,n} := \big(\theta'_{\sup} + \Delta^*_{a,n}\big) \vee 1$, where $\theta'_{\sup} := \theta_{\sup} \vee \theta$, and a constant $c > 0$ (given explicitly in the proof).

Proof. The proof is an adaptation of the proof of Arratia et al. (2003, Theorem 5.2). We fix an n with the required properties, and we set $p_k := \mathbb{P}[T_{a,n} = k]$ for $k \in \mathbb{N}$. Then the $\mathfrak{Z}$-conditioning relation entails

$$\begin{aligned}
&d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C_1^{(n)},\dots,C_a^{(n)}),\mathscr{L}(Z_1,\dots,Z_a)\big)\\
&\quad= \sum_{k=1}^{n} \mathbb{P}[T_{0,a}=k]\frac{\big(\mathbb{P}[T_{0,n}=n]-p_{n-k}\big)^+}{\mathbb{P}[T_{0,n}=n]}\\
&\quad\le \underbrace{\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{l=0}^{\lfloor n/2\rfloor} \mathbb{P}[T_{0,a}=k]\mathbb{P}[T_{0,a}=l]\frac{\big(p_{n-l}-p_{n-k}\big)^+}{\mathbb{P}[T_{0,n}=n]}}_{A_1(a,n)}\\
&\qquad+\underbrace{\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{l=\lfloor n/2\rfloor+1}^{n} \mathbb{P}[T_{0,a_n}=k]\mathbb{P}[T_{0,a_n}=l]\frac{p_{n-l}}{\mathbb{P}[T_{0,n}=n]}}_{A_2(a,n)}\\
&\qquad+\underbrace{\sum_{k=\lfloor n/2\rfloor+1}^{n} \mathbb{P}[T_{0,a}=k]}_{A_3(a,n)}\,.
\end{aligned}$$

To estimate $A_1(a,n)$, note that

$$A_1(a,n) = \sum_{0\le k<l\le n/2} \mathbb{P}[T_{0,a}=k]\mathbb{P}[T_{0,a}=l]\frac{\big|p_{n-l}-p_{n-k}\big|}{\mathbb{P}[T_{0,n}=n]}\,.$$

A calculation using the definition of $\Delta_{k,l}(j)$ in (5.29), and (5.30), shows that

$$\begin{aligned}
&(n-k)\big|p_{n-l}-p_{n-k}\big|\\
&\quad\le \theta\sum_{i=n-l-a}^{n-k-a-1}\mathbb{P}[T_{a,n}=i]+(l-k)\mathbb{P}[T_{a,n}=n-l]\\
&\qquad+\big|\kappa_{a,n}(l)\big|+\big|\kappa_{a,n}(k)\big|+\Delta_{a,n}(n-l)+\Delta_{a,n}(n-k)\\
&\quad\le \theta(\theta_{\sup}+\Delta^*_{a,n})\,\frac{l-k}{n-l-a}+(\theta_{\sup}+\Delta^*_{a,n})\,\frac{l-k}{n-l}\\
&\qquad+\big|\kappa_{a,n}(l)\big|+\big|\kappa_{a,n}(k)\big|+2\Delta^*_{a,n}\,,
\end{aligned}$$

with

$$\kappa_{a,n}(j) := \sum_{i=a+1}^{n}(\theta_i-\theta)\mathbb{P}[T_{a,n}+i=n-j]\qquad\text{for } j\in\mathbb{Z}_+.$$

Thus,

$$
\begin{aligned}
&A_1(a,n) \\
&\quad \le \sum_{0\le k<l\le n/2} \frac{(l-k)\mathbb{P}[T_{0,a}=k]\mathbb{P}[T_{0,a}=l]}{(n-k)\mathbb{P}[T_{0,n}=n]}\Big(\frac{(\theta_{\sup}+\Delta^*_{a,n})^2}{n-l-a}+\frac{(\theta_{\sup}+\Delta^*_{a,n})}{n-l}\Big) \\
&\qquad + \sum_{0\le k<l\le n/2} \frac{\mathbb{P}[T_{0,a}=k]\mathbb{P}[T_{0,a}=l]}{(n-k)\mathbb{P}[T_{0,n}=n]}\Big(\big|\kappa_{a,n}(l)\big|+\big|\kappa_{a,n}(k)\big|\Big) \\
&\qquad + 2\Delta^*_{a,n} \sum_{0\le k<l\le n/2} \frac{\mathbb{P}[T_{0,a}=k]\mathbb{P}[T_{0,a}=l]}{(n-k)\mathbb{P}[T_{0,n}=n]}\,.
\end{aligned}
$$

Recalling that $0\le k<l\le n/2$ and $a\le n/4$, we obtain further

$$
\begin{aligned}
A_1(a,n) \le \underbrace{\frac{2}{n\mathbb{P}[T_{0,n}=n]}\sum_{l=1}^{\lfloor n/2\rfloor} l\mathbb{P}[T_{0,a}=l]\Big(\frac{4(\theta_{\sup}+\Delta^*_{a,n})^2}{n}+\frac{2(\theta_{\sup}+\Delta^*_{a,n})}{n}\Big)}_{A_{1,1}(a,n)} \\
+ \underbrace{\frac{4}{n\mathbb{P}[T_{0,n}=n]}\sum_{l=0}^{\lfloor n/2\rfloor}\mathbb{P}[T_{0,a}=l]\big|\kappa_{a,n}(l)\big|}_{A_{1,2}(a,n)} + \underbrace{\frac{4\Delta^*_{a,n}}{n\mathbb{P}[T_{0,n}=n]}}_{A_{1,3}(a,n)}\,.
\end{aligned}
$$

Concerning $A_{1,1}(a,n)$, we use $\mathbb{E}T_{0,a}\le\theta_{\sup}a$ to conclude that

$$
\begin{aligned}
A_{1,1}(a,n) &\le 4\theta_{\sup}(\theta_{\sup}+\Delta^*_{a,n})\big(1+2(\theta_{\sup}+\Delta^*_{a,n})\big)\frac{a}{n^2\mathbb{P}[T_{0,n}=n]} \\
&\le \frac{12\theta_{\sup}\lambda^2_{a,n}}{n\mathbb{P}[T_{0,n}=n]}\frac{a}{n}\,.
\end{aligned}
$$

As for $A_{1,2}(n)$, we use Lemma A.17 (ii) to bound $|\kappa_{a,n}(l)|$, and we invoke the independence of $T_{0,a}$ and $T_{a,n}$ which leads to

$$
\begin{aligned}
&\sum_{l=0}^{\lfloor n/2 \rfloor} \mathbb{P}[T_{0,a} = l] \big|\kappa_{a,n}(l)\big| \\
&\quad \le 2\theta'_{\sup} m^2 d_{\mathrm{TV}}\big(\mathscr{L}(T_{a,n}), \mathscr{L}(T_{a,n}+1)\big) + \tilde{\theta}_n(m,\theta) \\
&\qquad + \theta'_{\sup} \sum_{l=0}^{\lfloor n/2 \rfloor} \mathbb{P}\big[T_{0,a} = l,\, n-l-n \le T_{a,n} < n-l-\lfloor n/m \rfloor m\big] \\
&\qquad + \theta'_{\sup} \sum_{l=0}^{\lfloor n/2 \rfloor} \mathbb{P}\big[T_{0,a} = l,\, n-l-\lfloor a/m \rfloor m - m \le T_{a,n} < n-l-a\big] \\
&\quad \le 2\theta'_{\sup} m^2 d_{\mathrm{TV}}\big(\mathscr{L}(T_{a,n}), \mathscr{L}(T_{a,n}+1)\big) + \tilde{\theta}_n(m,\theta) \\
&\qquad + \theta'_{\sup}\mathbb{P}\big[T_{0,n} < n - \lfloor n/m \rfloor m\big] + \theta'_{\sup}\mathbb{P}\big[n - \lfloor a/m \rfloor m - m \le T_{0,n} < n-a\big] \,.
\end{aligned}
$$

We further have

$$
\mathbb{P}\big[T_{0,n} < n - \lfloor n/m \rfloor m\big] \le \mathbb{P}\big[T_{0,n} \le m\big] \le \prod_{i=m+1}^{n} \mathbb{P}[Z_i = 0] \,. \tag{5.32}
$$

To bound this further, let $\{Z_i^*\}_{i\in\mathbb{N}}$ be a sequence of independent Poisson distributed random variables with expectations $\mathbb{E}Z_i^* = \mathbb{E}Z_i = \theta_i/i$ for $i \in \mathbb{N}$. Note that $\varepsilon_{i1}(i\mathbb{E}Z_i^*, r_i)$ is well-defined and equal to $\varepsilon_{i1}(i\mathbb{E}Z_i, r_i)$ for evey $i \in \mathbb{N}$. It follows from (A.13), assumption (5.28) and because $\mathbb{E}Z_i \le \theta_{\sup}/i$ for all $i \in \mathbb{N}$, that

$$
\begin{aligned}
\sum_{i=1}^{\infty} d_{\mathrm{TV}}\big(\mathscr{L}(Z_i), \mathscr{L}(Z_i^*)\big) &\le \sum_{i=1}^{\infty} \frac{3\theta_{\sup}}{i}\Big(|\varepsilon_{i1}(\theta_i, r_i)| + \frac{\theta_{\sup}}{i}\Big) + 2\sum_{i=1}^{\infty}\Big(\frac{\theta_{\sup}}{i} - 1\Big)^+ \\
&< \infty \,.
\end{aligned}
$$

We conclude, because also $\mathbb{P}[Z_i^* = 0] \ge \mathrm{e}^{-\theta_{\sup}}$ for all $i \in \mathbb{N}$, that

$$
\sum_{i=1}^{\infty} \bigg|\frac{\mathbb{P}[Z_i = 0]}{\mathbb{P}[Z_i^* = 0]} - 1\bigg| \le \mathrm{e}^{\theta_{\sup}} \sum_{i=1}^{\infty} d_{\mathrm{TV}}\big(\mathscr{L}(Z_i), \mathscr{L}(Z_i^*)\big) < \infty \,.
$$

This entails that, uniformly for m and n,

$$
\begin{aligned}
\prod_{i=m+1}^{n} \frac{\mathbb{P}[Z_i = 0]}{\mathbb{P}[Z_i^* = 0]} &\le \prod_{i=1}^{\infty}\bigg(1 + \bigg|\frac{\mathbb{P}[Z_i = 0]}{\mathbb{P}[Z_i^* = 0]} - 1\bigg|\bigg) \\
&\le \exp\bigg(\sum_{i=1}^{\infty}\bigg|\frac{\mathbb{P}[Z_i = 0]}{\mathbb{P}[Z_i^* = 0]} - 1\bigg|\bigg) =: Q < \infty \,.
\end{aligned}
$$

Returning to (5.32), we now conclude that

$$\begin{aligned}\mathbb{P}\big[T_{0,n} < n - \lfloor n/m \rfloor m\big] &\le \prod_{i=m+1}^{n} \frac{\mathbb{P}[Z_i = 0]}{\mathbb{P}[Z_i^* = 0]} \prod_{i=m+1}^{n} \mathbb{P}[Z_i^* = 0] \\ &\le Q \prod_{i=m+1}^{n} \mathbb{P}[Z_i^* = 0] \\ &\le Q\Big(\frac{m+1}{n+1}\Big)^{\theta} \exp\Big|\sum_{i=m+1}^{n} \frac{\theta_i - \theta}{i}\Big| \\ &\le Q(2\chi)^{\theta'_{\mathrm{sup}}}\Big(\frac{m}{n}\Big)^{\theta - \tilde{\theta}_n(m,\theta)},\end{aligned}$$

with $\chi := \exp(\pi^2/6 + 2\log 2)$, from Lemma A.16 (iii).

Recall that $n \ge 2(a+m)$ by assumption. This entails $n - \lfloor a/m \rfloor m - m \ge n/2$. Now (5.30) leads to

$$\mathbb{P}\big[n - \lfloor a/m \rfloor m - m \le T_{0,n} \le n - a\big] \le 2(\theta_{\mathrm{sup}} + \Delta^*_{a,n})\frac{m}{n} \le 2\lambda_{a,n}\frac{m}{n}.$$

Then

$$\begin{aligned}A_{1,2}(a,n) \le \frac{4}{n\mathbb{P}[T_{0,n} = n]}\bigg(&2\theta'_{\mathrm{sup}} m^2 d_{\mathrm{TV}}\big(\mathscr{L}(T_{a,n}), \mathscr{L}(T_{a,n}+1)\big) + \tilde{\theta}_n(m,\theta) \\ &+ Q(2\chi)^{\theta'_{\mathrm{sup}}}\Big(\frac{m}{n}\Big)^{\theta - \tilde{\theta}_n(m,\theta)} + 2\lambda_{a,n}\frac{m}{n}\bigg).\end{aligned}$$

It remains to give estimates for $A_2(a,n)$ and $A_3(a,n)$. Starting with

$$A_2(a,n) \le \sum_{l=\lfloor n/2 \rfloor + 1}^{n} \mathbb{P}[T_{0,a} = l]\frac{\mathbb{P}[T_{a,n} = n - l]}{\mathbb{P}[T_{0,n} = n]},$$

we bound $A_2(a,n)$ further in two ways. For the first, recall that $a \le n/4 < n/2 \le l$, by assumption. Then (5.31) implies

$$\sum_{l=\lfloor n/2 \rfloor + 1}^{n} \mathbb{P}[T_{0,a} = l]\frac{\mathbb{P}[T_{a,n} = n - l]}{\mathbb{P}[T_{0,n} = n]} \le \frac{8\theta_{\mathrm{sup}}^2 a}{n^2\mathbb{P}[T_{0,n} = n]} + \frac{2\Delta^*_{0,a}}{n\mathbb{P}[T_{0,n} = n]}.$$

The second bound is given by

$$\begin{aligned}\sum_{l=\lfloor n/2 \rfloor + 1}^{n} \mathbb{P}[T_{0,a} = l]\frac{\mathbb{P}[T_{a,n} = n - l]}{\mathbb{P}[T_{0,n} = n]} &\le \mathbb{P}[T_{0,a} \ge n/2]\frac{\sup_{j\in\mathbb{Z}_+}\mathbb{P}[T_{a,n} = j]}{\mathbb{P}[T_{0,n} = n]} \\ &\le \frac{2\theta_{\mathrm{sup}}\sup_{j\in\mathbb{Z}_+}\mathbb{P}[T_{a,n} = j]a}{n\mathbb{P}[T_{0,n} = n]}.\end{aligned}$$

All in all we have

$$A_2(a,n) \le \frac{8\theta_{\sup}^2 a}{n^2\mathbb{P}[T_{0,n}=n]} + \frac{2(\theta_{\sup}\vee 1)\min\{\Delta_{0,a}^*,\, \sup_{j\in\mathbb{N}}\mathbb{P}[T_{a,n}=j]a\}}{n\mathbb{P}[T_{0,n}=n]}\,.$$

Finally, $A_3(n)$ can be bounded from above by

$$A_3(n) \le \frac{2\mathbb{E}T_{0,a}}{n} \le 2\theta_{\sup}\frac{a}{n}\,.$$

Collecting the bounds for $A_{1,1}(a,n)$, $A_{1,2}(a,n)$, $A_2(a,n)$ and $A_3(a,n)$, along with $A_{1,3}(a,n)$, proves the theorem. □

As a corollary, we obtain a direct generalization of Theorem 5.17, in which $a = a_n$ may tend to infinity as $n\to\infty$.

Corollary 5.20. *Let $\mathfrak{C} := \{(\Omega_n,\nu_n,C^{(n)})\}_{n\in\mathbb{N}}$ be a $(\theta,\mathfrak{r},\mathfrak{Z})$-quasi-logarithmic RDCS such that* (5.28) *holds. Let $\{a_n\}_{n\in\mathbb{N}}$ be a positive integer sequence which satisfies $\lim_{n\to\infty} a_n/n = 0$. It follows that*

$$\lim_{n\to\infty} d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C_1^{(n)},\dots,C_{a_n}^{(n)}),\mathscr{L}(Z_1,\dots,Z_{a_n})\big) = 0\,.$$

Proof. The sequence $\mathfrak{Z}$ by assumption satisfies condition $\mathsf{SUQLC}(\theta,\mathfrak{r})$. In particular, this entails

$$\lim_{n\to\infty} d_{\mathrm{TV}}\big(\mathscr{L}(T_{a_n,n}),\mathscr{L}(T_{a_n,n}+1)\big) = 0\,.$$

Moreover, Theorem 3.21 yields

$$\lim_{n\to\infty} n\mathbb{P}[T_{0,n}=n] = p_\theta(1) > 0\,.$$

It also follows that $\tilde\theta_n(m_n,\theta)\to 0$ for every positive integer sequence $\{m_n\}_{n\in\mathbb{N}}$, with $m_n\to\infty$ and $m_n = \mathrm{o}(n)$. We may choose such an $\{m_n\}_{n\in\mathbb{N}}$ growing slowy enough for

$$\lim_{n\to\infty} m_n^2 d_{\mathrm{TV}}\big(\mathscr{L}(T_{a_n,n}),\mathscr{L}(T_{a_n,n}+1)\big) = 0$$

to hold. The corollary follows, if we can show that

$$\lim_{n\to\infty}\Delta_{a_n,n}^* = \lim_{n\to\infty}\min\big\{\Delta_{0,a_n}^*,\, \sup_{j\in\mathbb{Z}_+}\mathbb{P}[T_{a_n,n}=j]a_n\big\} = 0\,. \tag{5.33}$$

To do so, we go back to the proof of Lemma 3.18. It follows from (3.28) that

$$\Delta_{a_n,n}^* \le \sup_{k\in\mathbb{N}}\big|K_{a,n}^{(1)}(\mathfrak{r},\mathbf{1}\{\cdot=k\})\big|\,,$$

with $K^{(1)}_{a,n}(\mathfrak{r}, \mathbf{1}\{\cdot = k\})$ defined in (3.10). At the end of the proof of Lemma 3.18, in (3.32), we have the bound

$$\sup_{k\in\mathbb{N}}\big|K^{(1)}_{a,n}(\mathfrak{r}, \mathbf{1}\{\cdot = k\})\big| \ll \\ \sup_{j\in\mathbb{Z}_+} \mathbb{P}\big[T_{a_n,n} = j\big] \sum_{i=a_n}^{l_n-1}\Big(\mu_i(\mathfrak{r}) \vee \frac{1}{i+1}\Big) + \Big(\mu_{l_n}(\mathfrak{r}) \vee \frac{1}{l_n+1}\Big)\,, \tag{5.34}$$

for any positive sequence $\{l_n\}_{n\in\mathbb{N}}$ with $l_n > l_0$ for all $n \in \mathbb{N}$, where l_0 is given in (3.26).

Under our condition $\mathsf{SUQLC}(\theta, \mathfrak{r})$ on $\mathfrak{Z}$ we have $\mu_i(\mathfrak{r}) \to 0$, and, from Corollary 3.20,

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}_+} \mathbb{P}\big[T_{a_n,n} = j\big] = 0\,.$$

We may choose $\{l_n\}_{n\in\mathbb{N}}$ such that $l_n \to \infty$ slowly enough for the right hand side of (5.34) to converge to 0. Hence, $\Delta^*_{a_n,n} \to 0$. But then also by the same argument, replacing a_n by 0 and then n by a_n, $\Delta^*_{0,a_n} \to 0$ as long as $a_n \to \infty$. If $\{a_n\}_{n\in\mathbb{N}}$ is bounded or if $a_n \to \infty$ slowly enough, then

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}_+} \mathbb{P}\big[T_{a_n,n} = j\big] a_n = 0\,.$$

Now (5.33) follows. □

Under some further restrictions, we obtain convergence rates for the total variation distance in Theorem 5.19.

Corollary 5.21. *Let $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ be a $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic RDCS. Let $\{a_n\}_{n\in\mathbb{N}}$ be a natural number sequence that satisfies $\lim_{n\to\infty} a_n/n = 0$. We make the assumption that, for some $\alpha_1, \alpha_2, \alpha_3 > 0$,*

$$\mu_i(\mathfrak{r}) \vee \frac{1}{ir_i} = \mathrm{O}\Big(\frac{1}{i^{\alpha_1}}\Big)\,, \tag{5.35}$$

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{a_n,n}), \mathscr{L}(T_{a_n,n}+1)\big) = \mathrm{O}\Big(\Big(\frac{a_n}{n}\Big)^{\alpha_2}\Big)\,, \tag{5.36}$$

$$\tilde{\theta}_n(m_n, \theta) = \mathrm{O}\Big(\frac{1}{m_n^{\alpha_3}}\Big) \quad \textit{for all positive } \{m_n\}_{n\in\mathbb{N}} \textit{ with } \lim_{n\to\infty} m_n = \infty. \tag{5.37}$$

Then we have

$$d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C^{(n)}_1, \ldots, C^{(n)}_{a_n}), \mathscr{L}(Z_1, \ldots, Z_{a_n})\big) = \mathrm{O}\Big(\Big(\frac{a_n}{n}\Big)^{\alpha}\Big)\,, \tag{5.38}$$

where

$$\alpha := \min\Big\{\alpha_3 \frac{\alpha_2 \wedge (\theta/2) \wedge 1}{2+\alpha_3}, \frac{(\alpha_1 \wedge 1)((\alpha_1 \wedge (\theta/(\theta \vee 1)))/4)^2}{2 + (\alpha_1 \wedge 1)(\alpha_1 \wedge (\theta/(\theta \vee 1)))/4}\Big\}\,. \tag{5.39}$$

Proof. First, note that (5.35) implies (5.28), so that we may invoke Theorem 5.19. From this theorem it is immediate, using arguments as in the proof of Corollary 5.20, that for every sequence $\{m_n\}_{n\in\mathbb{N}}$ of natural numbers with $m_n \to \infty$ and $m_n = \mathrm{o}(n)$,

$$\begin{aligned} &d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C_1^{(n)},\dots,C_{a_n}^{(n)}),\mathscr{L}(Z_1,\dots,Z_{a_n})\big) \\ &\quad \ll \frac{a_n}{n} + m_n^2\Big(\frac{a_n}{n}\Big)^{\alpha_2} + \frac{1}{m_n^{\alpha_3}} + \Big(\frac{m_n}{n}\Big)^{(\theta/2)\wedge 1} \\ &\qquad + \Delta^*_{a_n,n} + \min\big\{\Delta^*_{0,a_n},\ \sup_{j\in\mathbb{Z}_+}\mathbb{P}[T_{a_n,n}=j]a_n\big\}\,. \end{aligned} \tag{5.40}$$

Recall from Lemma 3.19 (ii) that, since (5.35) entails $\varepsilon_{i1}(\theta_i,r_i) = \mathrm{O}(1/i^{\alpha_1})$, we have $\sup_{j\in\mathbb{Z}_+}\mathbb{P}\big[T_{a_n,n}=j\big] = \mathrm{O}((a_n/n)^{\varepsilon})$ with $\varepsilon := (\alpha_1 \wedge (\theta/(\theta\vee 1)))/4$.

Recall the proof of Corollary 5.20, in particular (5.34) and the inequality before that. We have

$$\Delta^*_{a_n,n} \ll \sup_{j\in\mathbb{Z}_+}\mathbb{P}\big[T_{a_n,n}=j\big]\sum_{i=a_n}^{l_n}\frac{1}{i^{\alpha_1\wedge 1}} + \frac{1}{l_n^{\alpha_1\wedge 1}}\,, \tag{5.41}$$

for any positive sequence $\{l_n\}_{n\in\mathbb{N}}$ with $l_n > l_0$ for all $n \in \mathbb{N}$, and l_0 defined in (3.26). If we choose $l_n \asymp (n/a_n)^{\delta}$, with $\delta := \varepsilon/2$ if $\alpha_1 \geq 1$ and $\delta := \varepsilon$ if $0 < \alpha_1 < 1$, it follows that

$$\Delta^*_{a_n,n} = \mathrm{O}\Big(\Big(\frac{a_n}{n}\Big)^{(\alpha_1\wedge 1)\varepsilon/2}\Big)\,.$$

Similarly, we obtain

$$\begin{aligned} \min\big\{\Delta_{0,a_n},\sup_{j\in\mathbb{Z}_+}\mathbb{P}[T_{a_n,n}=j]a_n\big\} &\ll \Big(\frac{1}{a_n}\Big)^{(\alpha_1\wedge 1)\varepsilon/2} \wedge \Big(\Big(\frac{a_n}{n}\Big)^{\varepsilon}a_n\Big) \\ &\ll \Big(\frac{a_n}{n}\Big)^{(\alpha_1\wedge 1)\varepsilon^2/(2+(\alpha_1\wedge 1)\varepsilon)}\,. \end{aligned}$$

Thus, (5.40) reduces to

$$\begin{aligned} &d_{\mathrm{TV}}\big(\mathscr{L}(C_1^{(n)},\dots,C_{a_n}^{(n)}),\mathscr{L}(Z_1,\dots,Z_{a_n})\big) \\ &\qquad \ll m_n^2\Big(\frac{a_n}{n}\Big)^{\alpha_2} + \frac{1}{m_n^{\alpha_3}} + \Big(\frac{m_n}{n}\Big)^{(\theta/2)\wedge 1} + \Big(\frac{a_n}{n}\Big)^{(\alpha_1\wedge 1)\varepsilon^2/(2+(\alpha_1\wedge 1)\varepsilon)}\,. \end{aligned}$$

The corollary follows, if we choose $m_n \asymp (n/a_n)^{\beta}$, with $\beta := (\alpha_2 \wedge (\theta/2) \wedge 1)/(2+\alpha_3)$. $\square$

Remark 5.22. We keep up the assumptions of Corollary 5.21. In various special situations, the exponent α in (5.39) has a simpler form.

(i) Let $\mathfrak{C}$ be a classical RDCS, such as an assembly, a multiset or a selection, as described in Subsection 4.1.4. In this case we have $\alpha_1 \geq 1$ in (5.35). This follows from Lemmas 2.5, 2.6 and 2.7, respectively; in case of multisets also recall Lemma 4.11. Note that assumption (5.28) holds automatically if $\alpha_1 \geq 1$. We have

$$\alpha = \min\left\{\alpha_3 \frac{\alpha_2 \wedge (\theta/2) \wedge 1}{2+\alpha_3}, \frac{((\theta/(\theta \vee 1))/4)^2}{2+(\theta/(\theta \vee 1))/4}\right\}.$$

(ii) Assume that $\mathfrak{C}$ is $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic with a sequence $\mathfrak{Z}$ of Poisson distributed random variables. Here we have $\Delta^*_{0,a_n} = \Delta^*_{a_n,n} = 0$ for all $n \in \mathbb{N}$, and thus the last two summand of (5.40) are equal to 0. The exponent α reduces to

$$\alpha = \alpha_3 \frac{\alpha_2 \wedge (\theta/2) \wedge 1}{2+\alpha_3}.$$

(iii) Assume that $\theta_i = i\mathbb{E}Z_i$, $i \in \mathbb{N}$, has a rational period. If q denotes the smallest integer multiple of the period, we may choose $m_n := q$ for all $n \in \mathbb{N}$ (instead of $m_n \asymp (n/a_n)^\beta$) at the end of proof of Corollary 5.21. With this choice, (5.37) holds in the form $\tilde{\theta}_n(m_n, \theta) = 0$ for all $n \in \mathbb{N}$. In this case, we have $(m_n/n)^{(\theta-\tilde{\theta}_n(m_n,\theta))\wedge 1} = (m_n/n)^{\theta\wedge 1} \ll (1/n)^{\theta \wedge 1}$ instead of $(m_n/n)^{(\theta-\tilde{\theta}_n(m_n,\theta))\wedge 1} \ll (m_n/n)^{(\theta/2)\wedge 1}$, which we have used in (5.40). This leads to

$$d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C_1^{(n)}, \ldots, C_{a_n}^{(n)}), \mathscr{L}(Z_1, \ldots, Z_{a_n})\big) \\ \ll \left(\frac{a_n}{n}\right)^{\alpha_2} + \left(\frac{1}{n}\right)^{\theta\wedge 1} + \left(\frac{a_n}{n}\right)^{(\alpha_1\wedge 1)\varepsilon^2/(2+(\alpha_1\wedge 1)\varepsilon)}.$$

We can set

$$\alpha := \min\left\{\alpha_2 \wedge \theta \wedge 1, \frac{(\alpha_1 \wedge 1)((\alpha_1 \wedge (\theta/(\theta \vee 1)))/4)^2}{2+(\alpha_1 \wedge 1)(\alpha_1 \wedge (\theta/(\theta \vee 1)))/4}\right\}.$$

(iv) Assume that $\theta_i = i\mathbb{E}Z_i$, $i \in \mathbb{N}$, is the integer skeleton of a function with irrational period p and finite variation over $[0, p]$, where $1/p$ is of finite irrationality type $\eta \geq 1$. Then $\alpha_3 = \eta$ (cf. Lemma 2.21). It follows that

$$\alpha = \min\left\{\eta \frac{\alpha_2 \wedge (\theta/2) \wedge 1}{2+\eta}, \frac{(\alpha_1 \wedge 1)((\alpha_1 \wedge (\theta/(\theta \vee 1)))/4)^2}{2+(\alpha_1 \wedge 1)(\alpha_1 \wedge (\theta/(\theta \vee 1)))/4}\right\}.$$

We also consider two more concrete examples, where, in particular (5.36) holds true with an explicit α_2.

Example 5.23. Let $\{\theta_i\}_{i\in\mathbb{N}}$ be a positive sequence with integer period p. Let $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ be a hybrid $\mathrm{ESF}(\theta_1, \ldots, \theta_p)$-structure (as introduced in Definition 4.14).

Here, the assumptions of Corollary 5.21 are satisfied. Moreover, each of the cases (i), (ii) and (iii) of Remark 5.22 holds. Thus, we have

$$\alpha = \alpha_2 \wedge \theta \wedge 1\,,$$

with $\theta := \frac{1}{p}\sum_{i=1}^{p}\theta_i$, and where α_2 can be derived from Theorem 6.18. Indeed, with $Y_i := iZ_i$ and $Z_i \sim \mathrm{Po}(\theta_i/i)$, we have

$$\mathbb{P}[Y_i = 0] = \mathrm{e}^{-\theta_i/i} \geq \mathrm{e}^{-\theta_{\sup}} \qquad \text{and} \qquad \mathbb{P}[Y_i = i] = \frac{\theta_i}{i}\mathrm{e}^{-\theta_i/i} \geq \frac{\theta_{\inf}}{i}\mathrm{e}^{-\theta_{\sup}}\,,$$

with $\theta_{\inf} := \min\{\theta_1, \ldots, \theta_p\}$, and we set $\psi_0 := \mathrm{e}^{-\theta_{\sup}}$ and $\psi_1 := \theta_{\inf}\mathrm{e}^{-\theta_{\sup}}$. Moreover, in view of Example 6.11, we can replace the expression $4R$ by 2 in Theorem 6.18. This leads to

$$\alpha_2 = \frac{\psi_0\psi_1}{4 + \psi_0\psi_1} = \frac{\theta_{\inf}}{\theta_{\inf} + 4\mathrm{e}^{2\theta_{\sup}}} \leq \theta \wedge 1\,,$$

and we obtain

$$\alpha = \frac{\theta_{\inf}}{\theta_{\inf} + 4\mathrm{e}^{2\theta_{\sup}}}\,. \tag{5.42}$$

Example 5.24. Let $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ be an additive arithmetic semigroup as introduced in Subsection 4.3.1. We assume that the counting function $a(n)$ of A satisfies a "Beurling-type" condition, as considered by Zhang (1996a, 1996b), of the form

$$a(n) = q^n \sum_{j=1}^{r} c_j n^{\rho_j - 1} + \mathrm{O}\big(q^n n^{-\delta}\big)\,,$$

for $r \in \mathbb{N}$, real numbers $\rho_1 < \cdots < \rho_r$ and $c_1, \ldots, c_r$ such that $\rho_r > 0$ and $c_r > 0$, $q > 1$ and $\delta > 2$ (see also Subsection 4.3.2). Under our restriction of $\delta > 2$, Theorem 4.32 and Definition 4.29 imply that the associated multiset $\mathfrak{C}_{\mathcal{A}} := \{(A(n), \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ is $(\theta, 1/q)$-quasi-logarithmic, with $\theta := \rho_r$. We therefore are in case (i) of Remark 5.22 and can choose $\alpha_1 \geq 1$ in (5.35).

The conditioning relation holds with independent $Z_i(1/q) \sim \mathrm{NB}(p(i), q^{-i})$, $i \in \mathbb{N}$, and it follows from the proof of Theorem 4.32 (in particular, cf. (4.37) and (4.40)) that

$$i\mathbb{E}Z_i(1/q) = \frac{ip(i)q^{-i}}{1 - q^{-i}} = u_i + \mathrm{o}(1)\,,$$

where $\{u_i\}_{i\in\mathbb{N}}$ is the integer skeleton of a sinusoidal function as in Lemma 2.22, which allows us to set $\alpha_3 := 1$ in (5.37).

It also follows from the proof Theorem 4.32 that $\{u_i\}_{i\in\mathbb{N}}$ is not a function which has period 2 which is also equal to 0 on odd integers. But then we can invoke Theorem 6.4 to conclude that (5.36) holds true with some $0 < \alpha_2 < 1$.

The exact value of α_2 is tedious to determine; it depends on the sinusoidal function that gives rise to $\{u_i\}_{i\in\mathbb{N}}$. Both Theorems 6.18 and 6.27 may be involved. For Theorem 6.18, we need to determine g, ψ_0 and ψ_1 in Assumption 6.6. If Theorem 6.27 can be applied, we can choose ψ_0 and ψ_1 via (4.38) (e. g. with $\varepsilon := 1/2$).

In any case, we can choose α_2 to be smaller than $\theta/2 = \rho_r/2$. This leads to

$$\alpha = \min\left\{\frac{\alpha_2}{3}, \frac{((\theta/(\theta\vee 1))/4)^2}{2+(\theta/(\theta\vee 1))/4}\right\}.$$

5.2.2.1 Related results from the literature

Our results on the distributional limits of the spectrum of small components of quasi-logarithmic structures are based on similar theorems by Arratia et al. (2003) proved in the context of logarithmic structures.

Theorem 5.17 is a direct generalization of Arratia et al. (2003, Theorem 5.1 and Theorem 6.5). However, the logarithmic counterpart of Theorem 5.21 by Arratia et al. (2003, Theorem 7.7) yields better convergence rates, namely

$$d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C_1^{(n)},\ldots,C_{a_n}^{(n)}),\mathscr{L}(Z_1,\ldots,Z_{a_n})\big) = \mathrm{O}\Big(\frac{a_n}{n}\Big),$$

under the conditions $(\mathbf{A}_0)$, $(\mathbf{D}_1)$ and $(\mathbf{B}_{11})$, described in Subsubsection 2.1.1.1. Refined total variation approximation results for logarithmic assemblies, multisets and selections can be found in Stark (1997a, 1997b, 1999).

In Theorems 5.17 and 5.21 the sequence $\theta_i = i\mathbb{E}Z_i$, $i\in\mathbb{N}$, is allowed to oscillate (due to the quasi-logarithmic condition), which is not the case in Arratia et al. (2003). An early result in this direction is due to Tarakanov and Čistjakov (1975). They proved an assertion as in Theorem 5.17 for the assembly of random permutations that do not contain any cycles of even length longer than a given $r\in\mathbb{N}$. In our language, this assembly is $(1/2,1)$-quasi-logarithmic and thus covered by Theorem 5.17. More generally, in Arratia et al. (1995, Theorem 2) a variant of Theorem 5.17 is proved for assemblies under the mild condition that

$$\sup_{i\in\mathbb{N}} i\mathbb{E}Z_i < \infty \qquad \text{and} \qquad \liminf_{i\in\mathbb{N}} i\mathbb{E}Z_i > 0\,. \tag{5.43}$$

Under our conditions, we have in particular $\operatorname{alim}_{i\to\infty} i\mathbb{E}Z_i$ (cf. (4.18)) and therefore (5.43) covers assemblies for which Theorem 5.17 does not apply. However, our theorem is valid in situations where $i\mathbb{E}Z_i = 0$ for infinitely many $i \in \mathbb{N}$.

Note that the convergence rate in (5.38) using the α in (5.42) of Example 5.23 does not seem to be optimal. Indeed, let us consider an $\mathrm{ESF}(\theta_1, \theta_2)$-structure. If $0 < \theta_2 < \theta_1$ and if $\theta_1 - \theta_2$ is an even integer it follows from Arratia et al. (1995, Corollary 4 and Lemma 6) that

$$d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C_1^{(n)}, \ldots, C_a^{(n)}), \mathscr{L}(Z_1, \ldots, Z_{a_n})\big) \asymp \frac{a_n}{n},$$

for $a_n = \mathrm{o}(n/\log n)$. Although our example gives a less accurate bound in this particular example, it covers $\mathrm{ESF}(\theta_1, \theta_2)$-structures for all $\theta_1, \theta_2 > 0$ and for all sequences $a_n = \mathrm{o}(n)$. In particular, any $\mathrm{ESF}(\theta_1, \theta_2)$-structure with $0 < \theta_1 < \theta_2$ is covered by our theorem, but not by Arratia et al. (1995, Theorem 3).

However, for $a_n = 1$, Arratia et al. (1995, Section 11) *conjecture* that if $0 < \theta_1 < \theta_2$ then it follows that

$$d_{\mathrm{TV}}\big(\mathscr{L}_{\nu_n}(C_1^{(n)}), \mathscr{L}(Z_1)\big) = \mathrm{O}\Big(\frac{1}{n^{\theta_1}}\Big).$$

The rate given by Example 5.23 is $\alpha = \theta_1/(\theta_1 + 4\mathrm{e}^{2\theta_2}) < \theta_1$.

5.2.3 Additive functions on AAS and on RDCS

In order to establish an analogue of the Kubilius main theorem, we first need to introduce the concept of an additive function on a RDCS. To do so, recall Definition 4.27 of an additive arithmetic semigroup $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$. Each function $f : A \to \mathbb{R}$ gives rise to a unique stochastic process as follows. For each $n \in \mathbb{N}$ such that the finite set $A(n) := \{u \in A : \|u\| = n\}$ is not empty, we consider the restriction $f_{|A(n)}$ of f to $A(n)$. We obtain a random variable

$$X^{(n)} := f_{|A(n)} : \big(A(n), \mathcal{P}(A(n)), \nu_n\big) \to \big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big), \tag{5.44}$$

with $\mathcal{P}(A(n))$ being the power σ-algebra, ν_n the uniform probability measure and $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra. If $A(n) = \emptyset$, we set $X^{(n)} := 0$.

If we have a sequence of $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$-valued random variables $\{X^{(n)}\}_{n\in\mathbb{N}}$, where $X^{(n)}$ is either defined on the probability space $(A(n), \mathcal{P}(A(n)), \nu_n)$ or $X^{(n)} = 0$, then a function $f : A \to \mathbb{R}$ is simply defined by

$$f(u) := X^{(n)}(u) \qquad \text{for every } u \in A(n) \text{ and every } A(n) \neq \emptyset.$$

Definition 5.25. Let $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ be an AAS. A function $f : A \to \mathbb{R}$ is said to have a *limiting distribution* if $\{X^{(n)}\}_{n\in\mathbb{N}}$ converges in distribution.

Here, we are interested in limiting distributions of additive functions on A. These functions are an analogue of the well-known class of additive functions on $\mathbb{N}$ which is studied in probabilistic number theory (see Tenenbaum (1995) for details).

Definition 5.26. Let $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ be an AAS. A function $f : A \to \mathbb{R}$ is *additive* if it satisfies

$$f(u \circ v) = u + v \qquad \text{for all coprime elements } u, v \in A.$$

The function f is *strongly additive* if it also satisfies

$$f(w^k) = f(w) \qquad \text{for all } w \in A \text{ and } k \in \mathbb{N}.$$

A typical example of an additive function is $\Omega : A \to \mathbb{R}$, where $\Omega(u)$ is the number of indecomposable elements, counted with multiplicities, in the decomposition of $u \in A$. This function is not strongly additive. An example of a strongly additive function is $\omega : A \to \mathbb{R}$, where $\omega(u)$ is the number of pairwise distinct indecomposable elements into which u can be decomposed. The function $\chi : A \to \mathbb{R}$, where $\chi(u)$ is the number of distinct sizes (i. e. values of the norm $\|\cdot\|$) of the indecomposable elements in the decomposition of u, is not additive unless there is at most one indecomposable element of a given size in P. If so, then $\chi = \omega$ is strongly additive. Also note that the norm $\|\cdot\|$ on A is additive (but not strongly additive) by definition.

In case of an additive function we can rewrite (5.44). Indeed, let $p_{1,i}, \ldots, p_{r_i,i}$ denote the r_i indecomposable elements of norm i in P, and assume that $u \in A(n)$ can be decomposed in the form

$$u = \prod_{i=1}^{n} \prod_{j=1}^{r_i} p_{j,i}^{m_{j,i}} \quad \text{with suitable } m_{j,i} = m_{j,i}(u) \in \mathbb{Z}_+.$$

Then we can write

$$X^{(n)} = \sum_{i=1}^{n} \mathbf{1}\{C_i^{(n)} \geq 1\} g_i \qquad \text{for all } n \in \mathbb{N}, \tag{5.45}$$

where $C_i^{(n)}(u) = m_{1,i} + \cdots + m_{r_i,i}$ is the number of irreducible elements of norm i in the decomposition of u, and where

$$g_i(u) = f\left(\prod\nolimits_{j=1}^{r_i} p_{j,i}^{m_{j,i}}\right).$$

Note that with the construction in (5.45), we not only cover additive functions, but also functions that are only "additive for irreducible elements of distinct norms"; that is, functions $f : A \to \mathbb{R}$ are covered that do not necessarily satisfy $f(p \circ q) = f(p) + f(q)$, for $p, q \in P$ with $p \neq q$ and $\|p\| = \|q\|$.

Now, we introduce a sequence $\{X^{(n)}\}_{n\in\mathbb{N}}$ of random variables that mimics additive functions on general random decomposable combinatorial structures as defined in Subsection 4.1.1. This construction can be found in Arratia et al. (2003, Section 8.5).

Let $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ be an RDCS. Here, we consider the sequence $\{C^{(n)}\}_{n\in\mathbb{N}}$ to be defined on some large *common probability space*. We consider mutually independent real-valued random variables $U_i(k)$, $i, k \in \mathbb{N}$, which are also independent of $C^{(n)}$ for every $n \in \mathbb{N}$, on the same probability space as $\{C^{(n)}\}_{n\in\mathbb{N}}$. Let $U_i(0) := 0$ for all $i \in \mathbb{N}$. Then the sequence

$$X^{(n)} := \sum_{i=1}^{n} \mathbf{1}\{C_i^{(n)} \geq 1\} U_i(C_i^{(n)}) , \qquad n \in \mathbb{N} , \tag{5.46}$$

corresponds to a real-valued function on the underlying RDCS, provided that the distributions of the $U_i(k)$ are suitably chosen. For example, if $U_i(k) := k$ for all $i, k \in \mathbb{N}$, then $f = \Omega$ counts the total number of components in $u \in A$. To model $f = \omega$, the function that counts the number of distinct components, the distributions of the $U_i(k)$ are more complicated. However, we can still choose $U_i(1) := 1$ for all $i \in \mathbb{N}$ in this case. If we finally set $U_i(k) := 1$ for all $i, k \in \mathbb{N}$, then we obtain $f = \chi$.

5.2.4 An analogue of the Kubilius main theorem for quasi-logarithmic RDCS

Following the definition in Subsection 4.2.1, let $\mathfrak{C} := \{(\Omega_n, \nu_n, C^{(n)})\}_{n\in\mathbb{N}}$ be a $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic RDCS for a constant $\theta > 0$, a positive integer sequence $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ and a sequence $\mathfrak{Z} := \{Z_i\}_{i\in\mathbb{N}}$ of independent $\mathbb{Z}_+$-valued random variables. We consider the sequence $\{C^{(n)}\}_{n\in\mathbb{N}}$, along with the set $\{U_i(k) : i, k \in \mathbb{N}\}$ of mutually independent random variables from the previous subsection, defined on some common probability space with $\{Z_j\}_{j\in\mathbb{N}}$. We also assume that $\{U_i(k) : i, k \in \mathbb{N}\}$ and $\{Z_j\}_{j\in\mathbb{N}}$ are independent. Let

$$U_i := U_i(1) \qquad \text{for all } i \in \mathbb{N}. \tag{5.47}$$

We introduce the centering and scaling constants

$$\mu(n) := \sum_{i=1}^{n} \mathbb{E}Z_i \mathbb{E}U_i \quad \text{and} \quad \sigma(n) := \left(\sum_{i=1}^{n} \mathbb{E}Z_i \mathbb{E}U_i^2 \right)^{1/2} , \tag{5.48}$$

and define, with $X^{(n)}$ as in (5.46),

$$\bar{X}^{(n)} := \frac{X^{(n)} - \mu(n)}{\sigma(n)} \qquad \text{for all } n \in \mathbb{N}.$$

Definition 4.17 entails that the sequence $\mathfrak{Z}$ satisfies condition $\mathsf{UC}(\mathfrak{r})$, which by Definition 2.2 means that $\lim_{i\to\infty} \mu_i(\mathfrak{r}) = 0$, with $\mu_i(\mathfrak{r})$ defined in (2.6). Here, we require a mild strengthening of this condition. We additionally require, as at the beginning of Subsection 5.2.2, that

$$\sum_{i=1}^{\infty} \frac{|\varepsilon_{i1}(i\mathbb{E}Z_i, r_i)|}{i} < \infty\,, \tag{5.49}$$

with $\varepsilon_i(i\mathbb{E}Z_i, r_i)$ defined in (2.3).

The following result is an analogue of the Kubilius main theorem from number theory (Kubilius, 1962; Elliott, 1980, Theorem 12.1) for quasi-logarithmic RDCS. Note that condition (5.50) corresponds to the assumption that an additive function belongs to the so-called *class H* (Kubilius, 1962; Zhang, 1996b).

Theorem 5.27. *Let $\mathfrak{C}$ be a $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic RDCS such that* (5.49) *holds. Assume that*

$$\lim_{n\to\infty} \sigma(n) = \infty \qquad \textit{and} \qquad \sigma(n) \textit{ is slowly varying at } \infty\,. \tag{5.50}$$

Then $\{\bar{X}^{(n)}\}_{n\in\mathbb{N}}$ converges in distribution as $n \to \infty$ if and only if there exists a distribution function $K : \mathbb{R} \to \mathbb{R}$, such that

$$\lim_{n\to\infty} \frac{1}{\sigma(n)^2} \sum_{i=1}^{n} \mathbb{E}Z_i \mathbb{E}\big\{U_i^2 \mathbf{1}\{U_i \le x\sigma(n)\}\big\} = K(x) \tag{5.51}$$

for all continuity points x of K. The limit then has the characteristic function

$$\psi(t) = \exp\bigg(\int_{-\infty}^{\infty} \big(\mathrm{e}^{itx} - 1 - itx\big) x^{-2} dK(x)\bigg)\,, \qquad t \in \mathbb{R}\,. \tag{5.52}$$

The proof is given in the next subsection. Here, we prove three rather straightforward corollaries. The first is an analogue of the Kubilius-Shapiro central limit theorem (Elliott, 1980, Theorem 12.2) in the context of quasi-logarithmic structures. Note that criterion (5.53) corresponds to the Lindeberg-Feller criterion in probability theory (Loève, 1977, p. 307).

Corollary 5.28. *Let $\mathfrak{C}$ be a $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic RDCS such that* (5.49) *holds. In order that $\{\bar{X}^{(n)}\}_{n\in\mathbb{N}}$ converges in distribution to* $\mathrm{N}(0,1)$ *it is sufficient that*

$$\lim_{n\to\infty} \frac{1}{\sigma(n)^2} \sum_{i=1}^{n} \mathbb{E}Z_i \mathbb{E}\big\{U_i^2 \mathbf{1}\{|U_i| \geq \varepsilon\sigma(n)\}\big\} = 0 \qquad \textit{for every } \varepsilon > 0. \tag{5.53}$$

If (5.50) *holds true, then* (5.53) *is also necessary.*

Proof. The standard normal distribution has characteristic function $\psi(t) = -t^2/2$, and therefore corresponds, via (5.52), to a function K in (5.51), which satisfies $K(x) = 1$ for $x > 0$ and $K(x) = 0$ for $x < 0$. In this case, (5.51) is equivalent to (5.53).

The corollary follows from Theorem 5.27 if we can show that (5.53) entails (5.50). Let $0 < \kappa < \lambda$. First, for every $\varepsilon > 0$, we have

$$\begin{aligned}
\left|1 - \frac{\sigma^2(\lfloor\kappa n\rfloor)}{\sigma^2(\lfloor\lambda n\rfloor)}\right| &\leq \frac{1}{\sigma^2(\lfloor\lambda n\rfloor)} \sum_{i=\lfloor\kappa n\rfloor+1}^{\lfloor\lambda n\rfloor} \mathbb{E}Z_i \mathbb{E}\big\{U_i^2 \mathbf{1}\{|U_i| < \varepsilon\sigma(\lfloor\lambda n\rfloor)\}\big\} \\
&\quad + \frac{1}{\sigma^2(\lfloor\lambda n\rfloor)} \sum_{i=\lfloor\kappa n\rfloor+1}^{\lfloor\lambda n\rfloor} \mathbb{E}Z_i \mathbb{E}\big\{U_i^2 \mathbf{1}\{|U_i| \geq \varepsilon\sigma(\lfloor\lambda n\rfloor)\}\big\} \\
&\leq \varepsilon^2 (\textstyle\sup_{i\in\mathbb{N}} i\mathbb{E}Z_i) \log\big(\lfloor\lambda n\rfloor / \lfloor\kappa n\rfloor\big) + \mathrm{o}(1)\,,
\end{aligned}$$

this last from (5.53), and recalling that $\sup_{i\in\mathbb{N}} i\mathbb{E}Z_i < \infty$, because $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ is a non-negative $\mathsf{A}(\theta)$-convergent sequence. Thus

$$\limsup_{n\to\infty} \left|1 - \frac{\sigma^2(\lfloor\kappa n\rfloor)}{\sigma^2(\lfloor\lambda n\rfloor)}\right| \leq c\varepsilon^2 \qquad \text{for every } \varepsilon > 0, \tag{5.54}$$

for some $c > 0$. Letting $\varepsilon \to 0$ implies that the left hand side of (5.54) converges to zero, which in turn implies that $\sigma(n)$ is slowly varying at infinity. Because (5.53) also entails $\sigma(n) \to \infty$, we have shown that (5.50) is satisfied. □

Now the analogue of the Erdős-Kac central limit theorem (Erdős and Kac, 1940; Elliott, 1980, Theorem 12.3) is immediate.

Corollary 5.29. *Let $\mathfrak{C}$ be a $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic RDCS such that* (5.49) *holds. Assume that $|U_i| \leq 1$ for all $i \in \mathbb{N}$ such that $\mathbb{E}Z_i > 0$. Then the sequence $\{\bar{X}^{(n)}\}_{n\in\mathbb{N}}$ converges in distribution to* $\mathrm{N}(0,1)$ *as $n \to \infty$.*

Corollary 5.30. *Let $\mathfrak{C}$ be a $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic RDCS such that* (5.49) *holds. Assume that $\{X^{(n)}\}_{n\in\mathbb{N}}$ models the function Ω which counts the number of components, or ω, which counts the distinct components, or χ, which counts the distinct sizes of the components. Then we have*

$$\frac{X^{(n)} - \theta \log n - \log \ell(n)}{\sqrt{\theta \log n}} \xrightarrow{n\to\infty} \mathrm{N}(0,1)\,, \tag{5.55}$$

where

$$\ell(n) := \exp\biggl(\sum_{i=1}^{n} \frac{i\mathbb{E}Z_i - \theta}{i}\biggr) \qquad n \in \mathbb{N}\,,$$

is slowly varying at infinity.

Proof. In any of the three cases, we have $U_i = 1$ for all $i \in \mathbb{N}$. Thus, we may apply Corollary 5.29. Because $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{A}(\theta)$, $\ell(n)$ is slowly varying at infinity (cf. Lemma 2.19). We obtain

$$\mu(n) = \sigma^2(n) = \sum_{i=1}^{n} \mathbb{E}Z_i = \sum_{i=1}^{n} \frac{\theta}{i} + \log \ell(n) \sim \theta \log n\,,$$

noting that $\log \ell(n)/\log n \to 0$ from (A.5). □

5.2.4.1 Related results from the literature

The analogue of the Kubilius main theorem for quasi-logarithmic structures is a generalization of a corresponding theorem for logarithmic structures in Arratia et al. (2003, Theorem 8.31) and Arratia et al. (2005). In fact, our method of proof in the next subsection mimics the arguments in Arratia et al. (2003).

Our result incorporates, at least to a great extent, a similar theorem of Zhang (1996b) in the context of additive arithmetic semigroups. More precisely, Zhang proves a Kubilius main theorem for additive arithmetic semigroups $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$, where the counting function $a(n)$ of A satisfies

$$a(n) = cq^n + \mathrm{O}(q^n n^{-2})\,, \tag{5.56}$$

for some constants $c > 0$ and $q > 1$ (cf. (4.30) from Subsection 4.3.2). Our Theorem 5.27 is satisfied under the Beurling-type condition (4.31), i. e.

$$a(n) = q^n \sum_{j=1}^{r} c_j n^{\rho_j - 1} + \mathrm{O}\bigl(q^n n^{-\delta}\bigr)\,,$$

if $\delta > 0$ satisfies the assumptions of Theorem 4.32; for then the AAS $\mathcal{A}$ is quasi-logarithmic. The only case which is covered by assumption (5.56) but not by the assumptions of Theorem 4.32 is where

$$a(n) = cq^n + s_n q^n n^{-2} \qquad \text{and} \qquad \sum_{n=0}^{\infty} a(n) z^n = 0 \quad \text{for } z = -1/q,$$

with constants $c > 0$ and $q > 1$ and a real-valued bounded sequence $\{s_n\}_{n\in\mathbb{N}}$ which is neither constant nor converges to zero more quickly than $n^{-\varepsilon}$ for any $\varepsilon > 0$. It is an open question, whether this situation is covered by the quasi-logarithmic condition and by our Theorem 5.27. Note that the result of Zhang only holds for strongly additive functions.

Zhang (2002) also proves a central limit theorem for the number χ of distinct sizes of the components in an AAS. Here, he assumes that $a(n) = cq^n + \mathrm{O}(q^n n^{-\delta})$ with $\delta > 2$, and his result is covered by Corollary 5.30 (though Zhang obtains also convergence rates in his theorem).

Recently, Wehmeier (2004) proved a version of the Erdős-Kac theorem for strongly additive functions on AAS $\mathcal{A}$ that satisfy

$$p(n) = \mathrm{O}\big(q^n/n\big) \quad \text{and} \quad a(n) = cq^n + \mathrm{O}\big(q^n (\log n)^{-k}\big) \quad \text{for every } k \in \mathbb{N}. \tag{5.57}$$

It remains open whether these assumptions are covered by our version of the Erdős-Kac theorem (Corollary 5.29). In fact, the remainder term of $a(n)$ is not small enough to be covered by the assumptions of Theorem 4.32 (and in turn by the assumptions of the prime element theorem of Zhang (1996a)) on which we rely to show that an AAS is quasi-logarithmic.

Flajolet and Soria (1990) prove central limit theorems similar to Corollary 5.28 in the context of assemblies, multisets and selections. Their assumptions allow the sequence $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ to oscillate, as does our quasi-logarithmic condition. Our version covers cases, even under the classical types of RDCS, for which the assumptions of Flajolet and Soria are not valid.

To show this, we recall Subsubsections 4.1.4.1 and 4.2.2.1, and we consider a (θ, x)-quasi-logarithmic assembly, $\theta > 0$ and $x > 0$, associated with a positive sequence $\{p(i)\}_{i\in\mathbb{N}}$. It can be shown with similar arguments as were used in the proof of Theorem 4.33 that the number $a(n)$ of instances of assemblies of size n constructed with $\{p(i)\}_{i\in\mathbb{N}}$ satisfies

$$a(n) \sim n! \, \frac{n^{\theta-1} x^{-n} \ell(n)}{\Gamma(\theta)} \,,$$

where

$$\ell(n) := \exp\Big(\sum_{i=1}^{n} \frac{p(i)x^i/(i-1)! - \theta}{i}\Big)$$

is slowly varying at infinity. We may choose $\{p(i)\}_{i\in\mathbb{N}}$ in a way such that $\ell(n)$ does not converge as $n \to \infty$. On the other hand, Flajolet and Soria (1990, Proposition 1) show that under their analytic conditions

$$a(n) \sim n!\,\frac{n^{\theta-1}x^{-n}\mathrm{e}^{K}}{\Gamma(\theta)}\,,$$

for some $K \in \mathbb{R}$.

5.2.5 Proof of the Kubilius main theorem

We keep the assumptions and notation from the previous subsection in force. Additionally we define, as earlier, $\theta_i := i\mathbb{E}Z_i$ for all $i \in \mathbb{N}$, $\theta_{\sup} := \sup_{i\in\mathbb{N}} \theta_i < \infty$, $\theta > 0$, $\theta'_{\sup} := \theta_{\sup} \vee \theta$ and

$$\tilde{\theta}_n(m,\theta) := \max_{0\le j\le \lfloor n/m\rfloor} \left| \frac{1}{m}\sum_{i=1}^{m} \theta_{jm+i} - \theta \right| \qquad \text{for all } m \in \mathbb{N}.$$

In Lemma 5.31 and Lemma 5.32 we prove technical bounds on probabilities such as $\mathbb{P}[C_i^{(n)} = k]$ and $\mathbb{P}[C_i^{(n)} \ge k]$. These bounds then allow us to mimic the proof of the Kubilius main theorem of Arratia et al. (2003, Theorem 8.31) for logarithmic structures.

Lemma 5.31. *Let $\mathfrak{C}$ be a $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic RDCS such that* (5.49) *holds. Then there is an $N \in \mathbb{N}$ and there is a constant $c > 0$, such that for every $n \ge i \ge N$, $k \in \mathbb{N}$ with $ik \le n$, and every $m \in \mathbb{N}$ with $m \le n$ we have*

$$\mathbb{P}[C_i^{(n)} = k] \le c m^{\theta'_{\sup}} \mathbb{P}[Z_i = k] \left(\frac{n - ik + 1}{n+1} \right)^{\theta - 1 - \tilde{\theta}_n(m,\theta)} . \tag{5.58}$$

Proof. The sequence $\mathfrak{Z} := \{Z_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{SUQLC}(\theta, \mathfrak{r})$. From (A.11) we have $\mathbb{P}[Z_i = 0] \to 1$; so choose i_0 large enough such that $\mathbb{P}[Z_i = 0] > 1/2$ for all $i \ge i_0$. The quasi-logarithmic condition, and in particular Theorem 3.21, implies that that there is a $n_0 \in \mathbb{N}$ such that $n\mathbb{P}[T_{0,n} = n] > p_\theta(1)/2$ and

$$\mathscr{L}\big(C^{(n)}\big) = \mathscr{L}\big(Z_1, \ldots, Z_n \,\big|\, T_{0,n} = n\big) \qquad \text{for all } n \ge n_0.$$

Fix $n \geq i \geq N := i_0 \vee n_0$. Invoking the independence of the random variables Z_j, $j \in \mathbb{N}$, we conclude that

$$\begin{aligned}\mathbb{P}[C_i^{(n)} = k] &= \frac{\mathbb{P}[Z_i = k]\mathbb{P}[T_{0,n} - iZ_i = n - ik]}{\mathbb{P}[T_{0,n} = n]} \\ &\leq \frac{\mathbb{P}[Z_i = k]}{\mathbb{P}[Z_i = 0]} \frac{\mathbb{P}[T_{0,n} = n - ik]}{\mathbb{P}[T_{0,n} = n]} \\ &\leq 2\mathbb{P}[Z_i = k] \frac{\mathbb{P}[T_{0,n} = n - ik]}{\mathbb{P}[T_{0,n} = n]} \,.\end{aligned} \tag{5.59}$$

To bound the quotient $\mathbb{P}[T_{0,n} = n-ik]/\mathbb{P}[T_{0,n} = n]$, let $\{Z_j^*\}_{j\in\mathbb{N}}$ be a sequence of independent Poisson distributed random variables with expectations $\mathbb{E}Z_j^* = \theta_j/j$ for $j \in \mathbb{N}$. Fix any $k \in \mathbb{N}$ such that $ik \leq n$. Then we can write

$$\begin{aligned}\frac{\mathbb{P}[T_{0,n} = n - ik]}{\mathbb{P}[T_{0,n} = n]} &= \underbrace{\prod_{j=n-ik+1}^{n} \mathbb{P}[Z_j^* = 0]}_{B_1(i,k,n)} \underbrace{\prod_{j=n-ik+1}^{n} \frac{\mathbb{P}[Z_j = 0]}{\mathbb{P}[Z_j^* = 0]}}_{B_2(i,k,n)} \\ &\quad \times \underbrace{\frac{\mathbb{P}[T_{0,n-ik} = n - ik]}{\mathbb{P}[T_{0,n} = n]}}_{B_3(i,k,n)} \,.\end{aligned} \tag{5.60}$$

Concerning $B_1(i,k,n)$, we conclude from (A.3) and Lemma A.16, (i) - (iii), that for any positive integers with $m \leq n$,

$$\begin{aligned}B_1(i,k,n) &= \exp\Big(-\sum_{j=n-ik+1}^{n} \frac{\theta}{j}\Big) \exp\Big(-\sum_{j=n-ik+1}^{n} \frac{\theta_j - \theta}{j}\Big) \\ &\leq \left(\frac{n-ik+1}{n+1}\right)^{\theta} \chi^{\theta'_{\sup}} m^{\theta'_{\sup}} \left(\frac{n}{(n-ik) \vee 1}\right)^{\tilde{\theta}_n(m,\theta)} \\ &\leq 2^{\theta'_{\sup}} \chi^{\theta'_{\sup}} m^{\theta'_{\sup}} \left(\frac{n-ik+1}{n+1}\right)^{\theta - \tilde{\theta}_n(m,\theta)} ,\end{aligned}$$

with $\chi := \exp(2 + \pi^2/6 + \log 2)$.

Since $\{Z_j\}_{j\in\mathbb{N}}$ satisfies condition $\mathsf{SUQLC}(\theta, \mathfrak{r})$, the sequence $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ by definition satisfies condition $\mathsf{A}(\theta)$, which in turn implies $\mathbb{E}Z_j = \mathrm{O}(1/j)$ (cf. Lemma 2.18). What is more, we have $\mathbb{E}Z_j^* = \mathbb{E}Z_j$ for all $j \in \mathbb{N}$ by construction. It then follows from (A.13) and assumption (5.49) (with the same arguments as used in the proof of Theorem 5.19, after inequality (5.32)) that uniformly in i, k and n

$$B_2(i,k,n) \le \exp\biggl(\sum_{j=1}^{\infty}\biggl|\frac{\mathbb{P}[Z_j=0]}{\mathbb{P}[Z_j^*=0]}-1\biggr|\biggr) =: Q < \infty\,.$$

Note that condition (5.49) is used to obtain this bound.

It remains to estimate $B_3(i,k,n)$. Recall that under our conditions on $\{Z_j\}_{j\in\mathbb{N}}$, $n\mathbb{P}[T_{0,n}=n]$ converges to $p_\theta(1)$ as $n\to\infty$, and that we have chosen n large enough for $n\mathbb{P}[T_{0,n}=n] > p_\theta(1)/2$ to hold at the beginning of this proof. It follows that

$$B_3(i,k,n) = \frac{\mathbb{P}[T_{0,n-ik}=n-ik]}{\mathbb{P}[T_{0,n}=n]} \le c'\,\frac{n+1}{n-ik+1}$$

for some constant $c'>0$.

The bounds of $B_1(i,k,n)$, $B_2(i,k,n)$ and $B_3(i,k,n)$ together with (5.60) and (5.59) lead to (5.58). $\square$

Lemma 5.32. *Let $\mathfrak{C}$ be a $(\theta,\mathfrak{r},\mathfrak{Z})$-quasi-logarithmic RDCS such that* (5.49) *holds. Let*

$$\eta_n(m) := 1-\theta+\tilde{\theta}_n(m,\theta) \qquad \textit{for all } m,n\in\mathbb{N}, \tag{5.61}$$

and

$$\delta_i := \delta_i(\theta_i,r_i) := \max\Bigl\{\bigl|\varepsilon_{i1}(\theta_i,r_i)\bigr|,\frac{1}{i}\Bigr\} \qquad \textit{for all } i\in\mathbb{N}. \tag{5.62}$$

Then there is a constant $d>0$, such that, for every $a,m,n\in\mathbb{N}$, with $a,m\le n$ and $\eta_n(m)\le 0$ we have

$$\mathbb{P}\Bigl[\bigcup\nolimits_{i=a+1}^{n}\{C_i^{(n)}\ge 2\}\Bigr] \le dm^{\theta'_{\sup}}\log(n/a)\sum_{i=a+1}^{\infty}\frac{\delta_i}{i}\,, \tag{5.63}$$

and for $a,m,n\in\mathbb{N}$, with $a,m\le n$ and $0<\eta_n(m)<1$

$$\mathbb{P}\Bigl[\bigcup\nolimits_{i=a+1}^{n}\{C_i^{(n)}\ge 2\}\Bigr] \le dm^{\theta'_{\sup}}\biggl(\log(n/a)\sum_{i=a+1}^{\infty}\frac{\delta_i}{i} + \frac{\log(n/a)}{1-\eta_n(m)}\bigl(\sup\nolimits_{j>\lfloor a/2\rfloor}\delta_j\bigr) + \frac{n^{\eta_n(m)}}{a}\bigl(\sup\nolimits_{j\ge a}\delta_j\bigr)\biggr). \tag{5.64}$$

Proof. Let $N \in \mathbb{N}$ be given as in Lemma 5.31. Let $n \geq a \geq N$ and $m \leq n$. Then we invoke (5.58) and conclude that

$$\begin{aligned}\mathbb{P}\Big[\bigcup\nolimits_{i=a+1}^{n}\{C_i^{(n)} \geq 2\}\Big] &\leq \sum_{k=2}^{\lfloor n/a \rfloor} \sum_{i=a+1}^{\lfloor n/k \rfloor} \mathbb{P}[C_i^{(n)} = k] \\ &\leq cm^{\theta'_{\sup}} \sum_{k=2}^{\lfloor n/a \rfloor} \sum_{i=a+1}^{\lfloor n/k \rfloor} \mathbb{P}[Z_i = k]\bigg(\frac{n-ik+1}{n+1}\bigg)^{-\eta_n(m)} .\end{aligned} \tag{5.65}$$

If $\eta_n(m) \leq 0$ we obtain from (5.65) and (A.12) that

$$\begin{aligned}\mathbb{P}\Big[\bigcup\nolimits_{i=a+1}^{n}\{C_i^{(n)} \geq 2\}\Big] &\leq cm^{\theta'_{\sup}} \sum_{k=2}^{\lfloor n/a \rfloor} \sum_{i=a+1}^{\lfloor n/k \rfloor} \mathbb{P}[Z_i = k] \\ &\leq cm^{\theta'_{\sup}} \sum_{k=2}^{\lfloor n/a \rfloor} \sum_{i=a+1}^{\lfloor n/k \rfloor} \frac{\mathbb{E}Z_i}{k}\big(\big|\varepsilon_{i1}(\theta_i, r_i)\big| + \mathbb{E}Z_i\big) ,\end{aligned} \tag{5.66}$$

As in the proof of Lemma 5.31 we have $\mathbb{E}Z_i = \mathrm{O}(1/i)$ under our conditions. This yields (5.63), the first part of the lemma.

Now, assume that $0 < \eta_n(m) < 1$. Here, we bound the ranges $a < i \leq \lfloor n/(2k) \rfloor$, $\lfloor n/(2k) \rfloor < i < \lfloor n/k \rfloor$ and $i = \lfloor n/k \rfloor$ in (5.65) separately. For the first range we obtain, once more using (A.12),

$$\begin{aligned}\sum_{k=2}^{\lfloor n/a \rfloor} \sum_{i=a+1}^{\lfloor n/(2k) \rfloor} \mathbb{P}[Z_i = k]\bigg(\frac{n+1}{n-ik+1}\bigg)^{\eta_n(m)} &\leq 2^{\eta_n(m)} \sum_{k=2}^{\lfloor n/a \rfloor} \sum_{i=a+1}^{\lfloor n/(2k) \rfloor} \mathbb{P}[Z_i = k] \\ &\leq c' \log(n/a) \sum_{i=a+1}^{\infty} \frac{\delta_i}{i} ,\end{aligned}$$

for some constant $c' > 0$.

In the second range we have

$$
\begin{aligned}
\sum_{k=2}^{\lfloor n/a\rfloor}\ \sum_{i=\lfloor n/(2k)\rfloor+1}^{\lfloor n/k\rfloor-1} & \mathbb{P}[Z_i=k]\biggl(\frac{n+1}{n-ik+1}\biggr)^{\eta_n(m)} \\
&\ll \sum_{k=2}^{\lfloor n/a\rfloor}\ \sum_{i=\lfloor n/(2k)\rfloor+1}^{\lfloor n/k\rfloor-1} \frac{\delta_i}{ik}\biggl(\frac{n+1}{\lfloor n/k\rfloor k-ik}\biggr)^{\eta_n(m)} \\
&\le \sum_{k=2}^{\lfloor n/a\rfloor}\ \sum_{i=\lfloor n/(2k)\rfloor+1}^{\lfloor n/k\rfloor-1} \frac{\delta_i}{(n/(2k))k}\biggl(\frac{n+1}{\lfloor n/k\rfloor k-ik}\biggr)^{\eta_n(m)} \\
&\ll n^{\eta_n(m)-1}\sum_{k=2}^{\lfloor n/a\rfloor}\frac{1}{k^{\eta_n(m)}}\sum_{i=\lfloor n/(2k)\rfloor+1}^{\lfloor n/k\rfloor-1}\delta_i\biggl(\frac{1}{\lfloor n/k\rfloor-i}\biggr)^{\eta_n(m)} \\
&\le n^{\eta_n(m)-1}\sum_{k=2}^{\lfloor n/a\rfloor}\frac{1}{k^{\eta_n(m)}}\bigl(\sup\nolimits_{j>\lfloor n/(2k)\rfloor}\delta_j\bigr)\sum_{i=1}^{\lfloor n/k\rfloor}\Bigl(\frac{1}{i}\Bigr)^{\eta_n(m)} \\
&\le \frac{n^{\eta_n(m)-1}}{1-\eta_n(m)}\bigl(\sup\nolimits_{j>\lfloor a/2\rfloor}\delta_j\bigr)\sum_{k=2}^{\lfloor n/a\rfloor}\frac{1}{k^{\eta_n(m)}}\Bigl(\frac{n}{k}\Bigr)^{1-\eta_n(m)} \\
&\le \frac{1}{1-\eta_n(m)}\log(n/a)\bigl(\sup\nolimits_{j>\lfloor a/2\rfloor}\delta_j\bigr)\,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{k=2}^{\lfloor n/a\rfloor}\mathbb{P}[Z_{\lfloor n/k\rfloor}=k]\biggl(\frac{n+1}{n-\lfloor n/k\rfloor k+1}\biggr)^{\eta_n(m)} &\ll n^{\eta_n(m)-1}\sum_{k=2}^{\lfloor n/a\rfloor}\delta_{\lfloor n/k\rfloor} \\
&\le n^{\eta_n(m)}\frac{1}{a}\bigl(\sup\nolimits_{j\ge a}\delta_j\bigr)
\end{aligned}
$$

Adding up these bounds, we obtain (5.64). □

Corollary 5.33. *Let $\mathfrak{C}$ be a $(\theta,\mathfrak{r},\mathfrak{Z})$-quasi-logarithmic RDCS such that* (5.49) *holds. Then for every large enough sequence $\{a_n\}_{n\in\mathbb{N}}$ of natural numbers that still satisfies $a_n\le n$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}a_n/n=0$, we have*

$$
\lim_{n\to\infty}\mathbb{P}\Bigl[\bigcup\nolimits_{i=a_n+1}^{n}\bigl\{C_i^{(n)}\ge 2\bigr\}\Bigr]=0\,. \tag{5.67}
$$

Proof. Under our assumption we have $\tilde{\theta}_n(m_n,\theta)\to 0$ for every positive integer sequence $\{m_n\}_{n\in\mathbb{N}}$ that satisfies $m_n\to\infty$ and $m_n=\mathrm{o}(n)$. It follows from (5.61) that $\eta_n(m_n)\to 1-\theta$ for any such sequence $\{m_n\}_{n\in\mathbb{N}}$.

Assume that $0 < \theta < 1$, and thus that $0 < 1-\theta+\theta/2 < 1$. In view of (5.64), we choose $\{a_n\}_{n\in\mathbb{N}}$, with $a_n \le n$ and $a_n = \mathrm{o}(n)$, growing fast enough for

$$f_n(a_n) := \log(n/a_n) \sum_{i=a_n+1}^{\infty} \frac{\delta_i}{i} + \frac{\log(n/a_n)}{\theta/2}\big(\sup\nolimits_{j>\lfloor a_n/2\rfloor} \delta_j\big) + \frac{n^{1-\theta+\theta/2}}{a_n}\big(\sup\nolimits_{j\ge a_n} \delta_j\big)$$

converge to 0 as $n \to \infty$. Now choose $\{m_n\}_{n\in\mathbb{N}}$, $m_n \to \infty$ and $m_n = \mathrm{o}(n)$, such that

$$\lim_{n\to\infty} m_n^{\theta'_{\sup}} f_n(a_n) = 0\,.$$

We have $0 < \eta_n(m_n) < 1-\theta+\theta/2 < 1$, and thus also $1-\eta_n(m_n) > \theta/2$, for all n large enough. Then we may invoke (5.64) to conclude (5.67).

Assume that $\theta > 1$. In view of (5.63), we choose $\{a_n\}_{n\in\mathbb{N}}$, with $a_n \le n$ and $a_n = \mathrm{o}(n)$, growing fast enough for

$$g_n(a_n) := \log(n/a_n) \sum_{i=a_n+1}^{\infty} \frac{\delta_i}{i}$$

to converge to 0 as $n \to \infty$. We choose $\{m_n\}_{n\in\mathbb{N}}$, $m_n \to \infty$ and $m_n = \mathrm{o}(n)$, such that

$$\lim_{n\to\infty} m_n^{\theta'_{\sup}} g_n(a_n) = 0\,.$$

Because of $\theta > 1$, $\eta_n(m_n) \le 0$ for all n large enough, and (5.63) entails (5.67).

Finally, let $\theta = 1$. Again we choose $\{a_n\}_{n\in\mathbb{N}}$, with $a_n \le n$ and $a_n = \mathrm{o}(n)$, such that $f_n(a_n) \to 0$, and we pick $\{m_n\}_{n\in\mathbb{N}}$, $m_n \to \infty$ and $m_n = \mathrm{o}(n)$, such that $m_n^{\theta'_{\sup}} f_n(a_n) \to 0$. Then we also have $m_n^{\theta'_{\sup}} g_n(a_n) \to 0$.

Moreover, $0 \le \eta_n(m_n) < 1$ for all n large enough. We obtain (5.67) from (5.63) and (5.64). □

Having established Lemma 5.31, Lemma 5.32 and Corollary 5.33, we may prove Theorem 5.27. There remain only minor differences between our proof and that of the corresponding theorem for logarithmic structures in Arratia et al. (2003, Theorem 8.31). We therefore give only an outline of the main ideas, providing details where the results above are involved. Recalling the definitions of U_i and $\sigma(n)$ in (5.47) and (5.48), respectively, we define for $0 \le j \le k \le n$

$$W_{j,k}^{(n)} := \sigma^{-1}(n) \sum_{i=j+1}^{k} \mathbf{1}\{C_i^{(n)} \ge 1\} U_i(C_i^{(n)})\,,$$

$$\widetilde{W}_{j,k}^{(n)} := \sigma^{-1}(n) \sum_{i=j+1}^{k} \mathbf{1}\{C_i^{(n)} = 1\} U_i\,.$$

We keep up the assumption that $\mathfrak{C}$ is a $(\theta, \mathfrak{r}, \mathfrak{Z})$-quasi-logarithmic RDCS such that (5.49) holds.

Lemma 5.34. *There exists an integer sequence $\{a_n\}_{n\in\mathbb{N}}$ with $1 \le a_n \le n$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} a_n/n = 0$, such that $\{W^{(n)}_{0,n}\}_{n\in\mathbb{N}}$ and $\{W^{(n)}_{0,a_n}\}_{n\in\mathbb{N}}$ have equivalent asymptotic behaviour.*

Proof. The lemma follows from Slutsky's theorem if we can show that

$$W^{(n)}_{a_n,n} \xrightarrow{\mathcal{D}} 0 \qquad \text{as } n \to \infty, \tag{5.68}$$

for some sequence $\{a_n\}_{n\in\mathbb{N}}$ with the required properties. In view of Corollary 5.33 we have for any sequence $\{a_n\}_{n\in\mathbb{N}}$ with these properties that grows fast enough

$$d_{\mathrm{TV}}\Big(\mathscr{L}\big(W^{(n)}_{a_n,n}\big), \mathscr{L}\big(\widetilde{W}^{(n)}_{a_n,n}\big)\Big) \le \mathbb{P}\Big[\bigcup\nolimits_{i=a_n+1}^{n} \big\{C^{(n)}_i \ge 2\big\}\Big] \xrightarrow{n\to\infty} 0\,.$$

Therefore, (5.68) holds true as soon as

$$\widetilde{W}^{(n)}_{a_n,n} \xrightarrow{\mathcal{D}} 0 \qquad \text{as } n \to \infty,$$

which we prove by showing that the sequence converges in probability.

To do so, let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers, which satisfies $m_n \le n$ for all $n \in \mathbb{N}$, $m_n = \mathrm{o}(n)$ and $m_n \to \infty$. Also fix an $\varepsilon > 0$. Invoking Lemma 5.31 with $m := m_n$, it follows for any integer sequence $\{l_n\}_{n\in\mathbb{N}}$, where $n/2 \le l_n \le n$ for all $n \in \mathbb{N}$, that

$$\begin{aligned}
\mathbb{P}\Big[\big|\widetilde{W}^{(n)}_{a_n,n}\big| \ge \varepsilon\Big] &\le \mathbb{P}\Big[\sigma^{-1}(n)\sum\nolimits_{i=a_n+1}^{l_n} \mathbf{1}\{C^{(n)}_i = 1\}|U_i| \ge \varepsilon\Big] \\
&\qquad + \mathbb{P}\Big[\sigma^{-1}(n)\sum\nolimits_{i=l_n+1}^{n} \mathbf{1}\{C^{(n)}_i = 1\}|U_i| \ne 0\Big] \\
&\le \frac{1}{\varepsilon\sigma(n)} \sum_{i=a_n+1}^{l_n} \mathbb{P}[C^{(n)}_i = 1]\mathbb{E}|U_i| + \sum_{i=l_n+1}^{n} \mathbb{P}[C^{(n)}_i = 1] \\
&\ll \frac{m_n^{\theta'_{\mathrm{sup}}}}{\sigma(n)} \underbrace{\sum_{i=a_n+1}^{l_n} \mathbb{P}[Z_i = 1]\bigg(\frac{n-i+1}{n+1}\bigg)^{\theta-1-\tilde{\theta}^{(n)}_{\max}(m_n)} \mathbb{E}|U_i|}_{B_1(n)} \\
&\qquad + m_n^{\theta'_{\mathrm{sup}}} \underbrace{\sum_{i=l_n+1}^{n} \mathbb{P}[Z_i = 1]\bigg(\frac{n-i+1}{n+1}\bigg)^{\theta-1-\tilde{\theta}^{(n)}_{\max}(m_n)}}_{B_2(n)}\,.
\end{aligned} \tag{5.69}$$

Under our assumptions $\theta_i = i\mathbb{E}Z_i$, $i \in \mathbb{N}$, is $\mathsf{A}(\theta)$-convergent and it follows that $\tilde{\theta}^{(n)}_{\max}(m_n)$ converges to 0. Thus, $\eta_n(m_n) = 1 - \theta + \tilde{\theta}^{(n)}_{\max}(m_n)$ converges to $1 - \theta < 1$ as $n \to \infty$.

Let $n \in \mathbb{N}$ be large enough for $\eta_n(m_n) < 1$. If $\eta_n(m_n) \leq 0$, it follows that

$$B_1(n) \leq \sum_{i=a_n+1}^{l_n} \mathbb{P}[Z_i = 1]\mathbb{E}|U_i| \leq \sum_{i=a_n+1}^{l_n} \mathbb{E}Z_i\mathbb{E}|U_i| \,,$$
$$B_2(n) \leq \sum_{i=l_n+1}^{n} \mathbb{P}[Z_i = 1] \leq \frac{\theta_{\sup}(n - l_n)}{l_n} \leq \frac{2\theta_{\sup}(n - l_n)}{n} \,.$$

If $0 < \eta_n(m_n) < 1$, we have

$$B_1(n) \leq \frac{2n}{n - l_n} \sum_{i=a_n+1}^{l_n} \mathbb{E}Z_i\mathbb{E}|U_i| \,,$$
$$B_2(n) \leq \frac{\theta_{\sup}(n+1)^{\eta_n(m_n)}}{l_n} \sum_{i=1}^{n-l_n} \Big(\frac{1}{i}\Big)^{\eta_n(m_n)}$$
$$\ll \frac{1}{1 - \eta_n(m_n)} \Big(\frac{n - l_n}{n}\Big)^{\theta - \tilde{\theta}^{(n)}_{\max}(m_n)} \ll \Big(\frac{n - l_n}{n}\Big)^{\theta - \tilde{\theta}^{(n)}_{\max}(m_n)} \,.$$

Hence, it follows from (5.69) that

$$\mathbb{P}\Big[\big|\widetilde{W}^{(n)}_{a_n,n}\big| \geq \varepsilon\Big] \ll \frac{m_n^{\theta'_{\sup}}}{\sigma(n)} \Big(\frac{n}{n - l_n}\Big) \sum_{i=a_n+1}^{l_n} \mathbb{E}Z_i\mathbb{E}|U_i| + m_n^{\theta'_{\sup}} \Big(\frac{n - l_n}{n}\Big)^{(\theta/2)\wedge 1} \,. \tag{5.70}$$

From the Cauchy-Schwarz inequality, the fact that $\mathbb{E}Z_i = \mathrm{O}(1/i)$, and the definition of $\sigma(n)$ in (5.48) we conclude that

$$\begin{aligned} \frac{1}{\sigma(n)} \sum_{i=a_n+1}^{l_n} \mathbb{E}Z_i\mathbb{E}|U_i| &\leq \frac{1}{\sigma(n)} \Big(\sum_{i=a_n+1}^{l_n} \mathbb{E}Z_i\Big)^{1/2} \Big(\sum_{i=a_n+1}^{l_n} \mathbb{E}Z_i\mathbb{E}U_i^2\Big)^{1/2} \\ &\ll \Big(\log(n/a_n)\Big(1 - \frac{\sigma^2(a_n)}{\sigma^2(n)}\Big)\Big)^{1/2} =: s_n \,. \end{aligned} \tag{5.71}$$

Since $\sigma(n)$ is a non-decreasing function and slowly varying at infinity by assumption (5.50), there is a sequence $h_n = \mathrm{o}(n)$ such that $\sigma^2(h_n)/\sigma^2(n) \to 1$ as $n \to \infty$ (Knopfmacher and Zhang, 2001, Lemma 7.6.3). Thus, we can choose $a_n \geq h_n$, $n \in \mathbb{N}$, (and also large enough for (5.67) to hold) in such a way that $s_n \to 0$ as $n \to \infty$, but still $a_n = \mathrm{o}(n)$. If we then choose, for example,

$$l_n \sim n(1 - \sqrt{s_n}) \qquad \text{and} \qquad m_n^{\theta'_{\sup}} \asymp s_n^{-1/4} \wedge s_n^{-\theta/8} \,,$$

then (5.70) and (5.71) imply $\widetilde{W}^{(n)}_{a_n,n} \xrightarrow{\mathcal{P}} 0$. This proves the lemma. $\square$

Corollary 5.20 entails that $\{W^{(n)}_{0,a_n}\}_{n\in\mathbb{N}}$ and

$$\sigma^{-1}(n)\sum_{i=1}^{a_n}\mathbf{1}\{Z_i \geq 1\}U_i(Z_i)\,, \qquad n\in\mathbb{N}\,, \tag{5.72}$$

have equivalent asymptotic behaviour for every sequence $a_n = \mathrm{o}(n)$ of positive integers with $1 \leq a_n \leq n$.

Then we define a sequence $\{\hat{Z}_i\}_{i\in\mathbb{N}}$ of independent Bernoulli random variables with expectations $\mathbb{E}\hat{Z}_i = (\theta_i/i)\wedge 1$, and independent also of $\{U_i\}_{i\in\mathbb{N}}$, on the same probability space as the other random variables. A coupling argument as used by Arratia et al. (2003, p. 213) shows that the sequence (5.72) and

$$\sigma^{-1}(n)\sum_{i=1}^{n}\hat{Z}_iU_i\,, \qquad n\in\mathbb{N}\,, \tag{5.73}$$

have equivalent asymptotic behaviour, where $\{a_n\}_{n\in\mathbb{N}}$ can be chosen to be the same sequence as in Lemma 5.34. Recalling the definition of $\mu(n)$ in (5.48), this finally entails that

$$\bar{X}_n = \frac{W^{(n)} - \mu(n)}{\sigma(n)} \qquad \text{and} \qquad \frac{\sum_{i=1}^{n}\hat{Z}_iU_i - \mathbb{E}\{\sum_{i=1}^{n}\hat{Z}_iU_i\}}{\sigma(n)}\,, \qquad n\in\mathbb{N}$$

have equivalent asymptotic behaviour. To the latter sequence we may apply the bounded variances limit theorem (Loève, 1977, Theorem 22.2A), which yields a convergence criterion that can be transformed to (5.51). This proves Theorem 5.27. $\square$

6 The coupling

6.1 Introduction

Let $\mathfrak{Y} := \{Y_i\}_{i\in\mathbb{N}}$ be a sequence of mutually independent $\mathbb{Z}$-valued random variables. Let

$$T_{a,n} := \sum_{i=a+1}^{n} Y_i \qquad \text{for all integers } (a,n) \text{ with } 0 \le a < n. \tag{6.1}$$

Definition 6.1. Let $\psi_0, \psi_1 > 0$. A number $i \in \mathbb{N}$ is a (ψ_0,ψ_1)-*good index* of the sequence $\mathfrak{Y}$ if

$$\mathbb{P}[Y_i = 0] \ge \psi_0 \quad \text{and} \quad \mathbb{P}[Y_i = i] \ge \frac{\psi_1}{i}\,. \tag{6.2}$$

A pair $(i,j) \in \mathbb{N}^2$ is a (ψ_0,ψ_1)-*good pair* if $i < j$ and if both i and j are (ψ_0,ψ_1)-good indices.

In this chapter we introduce a coupling method which, provided that $\mathfrak{Y}$ has "not too few" (ψ_0,ψ_1)-good pairs, allows us to obtain bounds on the total variation distance between $\mathscr{L}(T_{a,n})$ and $\mathscr{L}(T_{a,n}+1)$ of the form

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{a,n}), \mathscr{L}(T_{a,n}+1)\big) \le c\Big(\frac{a}{n}\Big)^{\alpha}\,, \tag{6.3}$$

for some constants $c > 0$ and $0 < \alpha < 1$.

Remark 6.2. What we bear in mind with the distributional assumption on $\mathfrak{Y}$ are situations where

$$Y_i := iZ_i \qquad \text{for all } i \in \mathbb{N},$$

with $\{Z_i\}_{i\in\mathbb{N}}$ being a sequence of independent $\mathbb{Z}_+$-valued random variables that satisfies the uniform quasi-logarithmic condition $\mathsf{UQLC}(\theta,\mathfrak{r})$ for some $\theta > 0$ and $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ (cf. Definition 2.25). We then derive sufficent conditions for the smoothness condition SC from Definition 2.31 to hold.

Note that from Lemma 2.26 and Lemma A.13,

$$\lim_{i\to\infty} \mathbb{P}[Y_i = 0] = 1 \qquad \text{and} \qquad \mathbb{P}[Y_i = i] = \frac{i\mathbb{E}Z_i}{i} + \mathrm{o}(1/i)\,,$$

and $\operatorname{alim}_{i\to\infty} i\mathbb{E}Z_i = \theta$. In order to find "enough" (ψ_0, ψ_1)-good pairs, for some $\psi_0, \psi_1 > 0$, it is necessary that $i\mathbb{E}Z_i$ is bounded away from 0 for "not too few" $i \in \mathbb{N}$.

Recall that in Proposition 2.33 and in Lemma 2.34 we have already seen special situations, where $i\mathbb{E}Z_i$ is bounded away from 0 for all i large enough or for all $i \in \mathbb{N}$, respectively. In Theorem 6.4 we encounter a more complicated situation. Note that these results are based on Theorem 6.18 or Theorem 6.27, where the restriction of the Y_i to the form iZ_i, with Z_i satisfying the uniform quasi-logarithmic condition, is not required.

A very simple, yet typical, example of a sequence $\{Y_i\}_{i\in\mathbb{N}}$ that satisfies (6.2), with suitable ψ_0, ψ_1, for every $i \geq 2$ is the following.

Example 6.3. Assume that $Y_i := iZ_i$, where $Z_i \sim \mathrm{Be}(1/i)$ for all $i \in \mathbb{N}$. Then we have for every $i \geq 2$

$$\mathbb{P}[Y_i = 0] = 1 - \frac{1}{i} \geq \frac{1}{2} \quad \text{and} \quad \mathbb{P}[Y_i = i] \geq \frac{1}{i}\,.$$

Thus, choosing $\psi_0 := 1/2$ and $\psi_1 := 1$, every natural number $i \geq 2$ is a (ψ_0, ψ_1)-good index.

We consider a more interesting situation. For this, we define a sinusoidal function $\Theta : \mathbb{R} \to \mathbb{R}_+$ by

$$\Theta(t) := \theta + \sum_{l=1}^{k} \lambda_l \cos(2\pi f_l t - \varphi_l)\,, \qquad t \in \mathbb{R}\,, \tag{6.4}$$

with $\theta > 0$, amplitudes $\lambda_l > 0$ such that $\sum_{l=1}^{k} \lambda_l \leq \theta$, frequencies $f_l \in \mathbb{R}_+ \setminus \mathbb{Z}_+$ and phases $0 \leq \varphi_l < 2\pi$. Let $\{\theta_i\}_{i\in\mathbb{N}}$ be the integer skeleton of $\Theta(t)$. Note that the summands $\cos(2\pi f_l i - \varphi_l)$ are not constant as functions of $i \in \mathbb{N}$, because of our restriction of f_l to non-integers.

We emphasise two special cases, namely

$$\sum_{l=1}^{k} \lambda_l = \theta\,, \qquad f_l = 1/2\,, \qquad \varphi_l = \pi \qquad \text{for all } l = 1, \ldots, k, \tag{6.5a}$$

$$\sum_{l=1}^{k} \lambda_l = \theta\,, \qquad f_l = 1/2\,, \qquad \varphi_l = 0 \qquad \text{for all } l = 1, \ldots, k. \tag{6.5b}$$

If (6.5a) is satisfied, we have $\theta_i = 0$ for every even $i \in \mathbb{N}$. Under (6.5b) we have $\theta_i = 0$ for every odd $i \in \mathbb{N}$.

A consequence of the coupling developed in the next two sections is the following theorem.

Theorem 6.4. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables which satisfies the uniformity condition* $\mathsf{UC}(\mathfrak{r})$ *for some positive integer sequence $\mathfrak{r}$ (cf. Definition 2.2), and which satisfies*

$$i\mathbb{E}Z_i = \theta_i + \mathrm{o}(1)\,,$$

where $\{\theta_i\}_{i\in\mathbb{N}}$ is the integer skeleton of the sinusoidal function $\Theta(t)$ from (6.4).

If (6.5b) *does* not *hold, then $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition* $\mathsf{SUQLC}(\theta,\mathfrak{r})$. *In particular, if $Y_i := iZ_i$ for all $i \in \mathbb{N}$, we have, for some $c > 0$ and $0 < \alpha < 1$,*

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{a,n}),\mathscr{L}(T_{a,n}+1)\big) \le c\Big(\frac{a\vee 1}{n}\Big)^{\alpha}$$

for every $n \in \mathbb{N}$ and every non-negative integer $a < n$.

If (6.5b) does *hold, $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition* $\mathsf{UQLC}(\theta,\mathfrak{r})$. *Condition* SC *need not hold. In fact, if $i\mathbb{E}Z_i = \theta_i$ for all $i \in \mathbb{N}$, we have*

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{a,n}),\mathscr{L}(T_{a,n}+1)\big) = 1$$

for every $n \in \mathbb{N}$ and every non-negative integer $a < n$.

6.1.1 The Mineka coupling approach

Usually, the total variation distance between distributions of the form $\mathscr{L}(T_{a,n})$ and $\mathscr{L}(T_{a,n}+1)$ is examined using the Mineka coupling, developed independently by Mineka (1973) and Rösler (1977a, 1977b) for the analysis of sums of independent, not necessarily identically distributed random variables. We refer to Lindvall (2002, Section II.14) for the case of identically distributed random variables.

Barbour and Xia (1999, Proposition 4.6) showed that

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}),\mathscr{L}(T_{0,n}+1)\big) \le \Big(\sum\nolimits_{i=1}^{n} u_i\Big)^{-1/2}\,, \tag{6.6}$$

where

$$u_i := \frac{1}{2}\wedge\Big(1 - d_{\mathrm{TV}}\big(\mathscr{L}(Y_i),\mathscr{L}(Y_i+1)\big)\Big)\,.$$

The proof, which we sketch here briefly, is based on this coupling method. We choose a copy $\{Y_i'\}_{i\in\mathbb{N}}$ of $\{Y_i\}_{i\in\mathbb{N}}$ on the same probability space. Mineka coupling of $\{Y_i\}_{i\in\mathbb{N}}$ and $\{Y_i'\}_{i\in\mathbb{N}}$ gives rise to a simple symmetric random walk $\{V_n\}_{n\in\mathbb{Z}_+}$, where

$$V_0 := 0 \qquad \text{and} \qquad V_n := \sum_{i=1}^{n}\big(Y_i - Y_i'\big) \quad \text{for all } n \in \mathbb{N}.$$

If τ is the time at which $\{V_n\}_{n\in\mathbb{Z}_+}$ first hits level 1, the coupling inequality (Lindvall, 2002, Section I.2) and the reflection principle yield

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}), \mathscr{L}(T_{0,n}+1)\big) \le \mathbb{P}[\tau > n] = \mathbb{P}[V_n \in \{-1, 0\}]\,,$$

from which (6.6) can be deduced using Fourier inversion.

However, the inequality (6.6) gives slow convergence rates if the $\{Y_i\}_{i\in\mathbb{N}}$ that arises from a quasi-logarithmic sequence $\{Z_i\}_{i\in\mathbb{N}}$ as described in Remark 6.2. We illustrate this in the simple situation of Example 6.3, where $Y_i \sim i\mathrm{Be}(1/i)$. Here, the distributions of Y_i and Y_i+1 do not overlap, so that the total variation distance between these laws is one; hence (6.6) gives the useless bound

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}), \mathscr{L}(T_{0,n}+1)\big) \le \infty \qquad \text{for all } n \in \mathbb{N}.$$

This can be improved by building blocks of two random variables. That is, we consider new random variables $W_i := Y_{2i-1} + Y_{2i}$, instead of Y_i. Here,

$$d_{\mathrm{TV}}\big(\mathscr{L}(W_i), \mathscr{L}(W_i+1)\big) = 1 - \Big(1 - \frac{1}{2i-1}\Big)\frac{1}{2i} \qquad \text{for all } i \in \mathbb{N},\ i \ge 2,$$

and thus

$$\begin{aligned} d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}), \mathscr{L}(T_{0,n}+1)\big) &\le d_{\mathrm{TV}}\Big(\mathscr{L}\big(T_{0,2\lfloor n/2\rfloor}\big), \mathscr{L}\big(T_{0,2\lfloor n/2\rfloor}+1\big)\Big) \\ &\le \bigg(\sum_{i=1}^{\lfloor n/2\rfloor}\Big(1 - \frac{1}{2i-1}\Big)\frac{1}{2i}\bigg)^{-1/2} \\ &\asymp (\log n)^{-1/2}\,. \end{aligned}$$

A convergence rate of better order cannot be achieved in this way by more complicated blocking. Choosing random variables of the form $W_i := Y_i + Y_{i+j}$, where $j > 1$, is of no use, since the distributions of W_i and $W_i + 1$ do not overlap in this case unless $i = 1$. Also, considering arbitrary blocks of three or more random variables, $W_i := Y_i + Y_{i+j_1} + \cdots + Y_{i+j_k}$ say, where $1 \le j_1 < \cdots < j_k$, does not improve the convergence rate. The only contributions to $d_{\mathrm{TV}}(\mathscr{L}(W_i), \mathscr{L}(W_i+1))$ which have significant influence on the bound (6.6) arise by coupling events in which only one of the random variables $Y_i, Y_{i+j_1}, \ldots, Y_{i+j_k}$ takes the value 1. This leads, in the best case, to a bound of order $(\log n)^{-1/2}$, just as in the case of blocks of two random variables.

6.1.2 Sketch of an alternative approach

We now describe an alternative to the Mineka coupling which in the situation of Example 6.3 yields

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}), \mathscr{L}(T_{0,n}+1)\big) = \mathrm{O}\big(n^{-\alpha}\big)\,, \qquad \text{for some } 0 < \alpha < 1,$$

and therefore proves more useful than the Mineka coupling in the analysis of sums of independent random variables $\{Y_i\}_{i\in\mathbb{N}}$ that arise from a quasi-logarithmic sequence $\{Z_i\}$ as in Remark 6.2.

As before, we let $\{Y_i'\}_{i\in\mathbb{N}}$ be a copy of $\{Y_i\}_{i\in\mathbb{N}}$ on a common probability space, and we also invoke the coupling inequality

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}),\mathscr{L}(T_{0,n}+1)\big)\le\mathbb{P}[\tau>n]\,.$$

Here, τ is the time where a process $\{V_m\}_{m\in\mathbb{Z}_+}$,

$$V_m:=\sum_{k=0}^{m}U_k \qquad \text{for each } m\in\mathbb{Z}_+, \tag{6.7}$$

starting with $V_0=U_0:=0$, first hits level 1. The process $\{U_m\}_{m\in\mathbb{N}}$ is constructed with the following coupling. We choose an arbitrary $0<\varepsilon<1$. Then we define

$$l_n:=\Big\lfloor\varepsilon\,\frac{\log n}{\log 2}\Big\rfloor\asymp\log n\,,\quad L_n:=2^{l_n}\asymp n^{\varepsilon}\quad\text{and}\quad M_n:=\Big\lfloor\frac{n}{L_n}\Big\rfloor\asymp n^{1-\varepsilon}\,, \tag{6.8}$$

and consider blocks

$$B_1^{(n)},B_2^{(n)},\ldots,B_{M_n}^{(n)}$$

of consecutive integers having length L_n, starting with $B_1^{(n)}:=\{1,\ldots,L_n\}$.

We define couplings $(S_{i,0},S_{i,0}')$ of pairs $S_{i,0}:=Y_i+Y_{i+1}$ for the odd numbers $i\in B_1^{(n)}$, such that $S_{i,0}-S_{i,0}'\in\{-1,0,1\}$. The random variable U_1 is defined as the sum of such differences over the odd $i\in B_1^{(n)}$, and the coupling is defined in such a way that $U_1\in\{-1,0,1\}$ also.

If $U_1=1$, then we are done. If $U_1=0$, construct U_2 in the second block in the same way as U_1. If $U_1=-1$, we build $U_2\in\{-2,0,2\}$ using couplings of random variables of the form Y_i+Y_{i+2} in block $B_2^{(n)}$, such that the corresponding differences only take the values -2, 0 or 2. Again we have three possibilities. If $U_2=2$, we are done, if $U_2=0$, we repeat this coupling in the next block, and if $U_2=-2$, the definition of U_3 in the third block is based on random variables of the form Y_i+Y_{i+4}.

In general, if l U-jumps, i. e. events of the form $U_k\neq 0$, have occurred in the first $m-1$ blocks, and level 1 has still not been reached, then U_m is based on couplings of

$$S_{i,l}:=Y_i+Y_{i+2^l}\,,$$

with suitably chosen indices $i\in B_m^{(n)}$.

Level 1 is reached with high probability within the first M_n blocks. Indeed, in each block $B_m^{(n)}$ we have $L_n/2$ possibilities for a S-jump, i. e. an event of the form $S_{i,l} - S'_{i,l} \neq 0$, to occur, and for $i \geq 2$ it is possible to couple $W_{i,l}$ and $S'_{i,l}$ in such a way that

$$\mathbb{P}\big[|S_{i,l} - S'_{i,l}| = 2^l\big] \geq \frac{1}{i + 2^l} \geq \frac{1}{mL_n};$$

hence it is possible to construct a coupling in such a way that

$$\mathbb{P}[U_m \neq 0] = 1 - \left(1 - \frac{1}{mL_n}\right)^{L_n/2} \asymp \frac{1}{m}.$$

Thus, with our choice of $l_n \asymp \log n$ and $M_n \asymp n^{1-\varepsilon}$, it can be shown using Chernov bounds that the probability of having fewer than l_n U-jumps among the blocks $B_1^{(n)}, \ldots, B_{M_n}^{(n)}$ is of order $\mathrm{O}(n^{-\delta})$, for some $0 < \delta < 1$. But the probability that level 1 is not hit in any of the blocks $B_1^{(n)}, \ldots, B_{M_n}^{(n)}$, given that l_n (or more) jumps occurred is $1/2^{l_n}$ (or smaller). We thus obtain

$$\begin{aligned} d_{\mathrm{TV}}\big(\mathscr{L}(T_{0,n}), \mathscr{L}(T_{0,n}+1)\big) &\leq \mathbb{P}[\tau > M_n] \\ &\leq \mathbb{P}\big[\text{number of } U\text{-jumps is} < l_n\big] + \frac{1}{2^{l_n}} \\ &= \mathrm{O}\big(n^{-\delta} + n^{-\varepsilon}\big). \end{aligned}$$

The formal conditions for this coupling are given in the sections below. However, we will also consider more general conditions. Under these conditions, the restrictions on which random variables Y_i can be used for the coupling are substantially relaxed. For example, many of these can be zero almost surely and level 1 may not be directly reachable. In such cases, the process $\{V_m\}_{m\in\mathbb{Z}_+}$ may have to reach another specified level first, before the coupling is switched and level 1 can be hit.

6.2 A coupling inequality

We start with some notation which we use throughout this and the subsequent sections. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of integers with $0 \leq a_n < n$ for all $n \in \mathbb{N}$. For every $n, N \in \mathbb{N}$ we set

$$\tilde{a}_n := \tilde{a}_n(N) := a_n \vee (N-1) \in \mathbb{Z}_+ \quad \text{and} \quad \bar{a}_n := \bar{a}_n(N) := \tilde{a}_n(N) \vee 1 \in \mathbb{N}. \tag{6.9}$$

Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of natural numbers which will serve as block lengths as in Subsection 6.1.2. We want to couple $T_{a_n,n} = \sum_{i=a_n+1}^{n} Y_i$ and

$T_{a_n,n}+1$. In order to make weak assumptions on which indices i can be used, we construct the coupling so as to use only indices from some fixed N onwards; i. e. those that are larger than $\tilde{a}_n = \tilde{a}_n(N)$. We therefore define the blocks

$$B_m^{(n)} := \big\{(w_n+m-1)L_n+1,\, \ldots,\, (w_n+m)L_n\big\} \qquad \text{for all } m,n \in \mathbb{N}, \tag{6.10}$$

where

$$w_n := \left\lceil \frac{\tilde{a}_n}{L_n} \right\rceil \qquad \text{for all } n \in \mathbb{N}. \tag{6.11}$$

Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of mutually independent $\mathbb{Z}$-valued random variables. We set

$$T_{B_m^{(n)}} := \sum_{i\in B_m^{(n)}} Y_i \qquad \text{for all } m,n \in \mathbb{N}. \tag{6.12}$$

Let $\{T'_{B_m^{(n)}}\}_{m\in\mathbb{N}}$ and $\{T''_{B_m^{(n)}}\}_{m\in\mathbb{N}}$ be copies of the process $\{T_{B_m^{(n)}}\}_{m\in\mathbb{N}}$ defined on a common probability space. We define

$$V_0^{(n)} := 0 \qquad \text{and} \qquad V_m^{(n)} := \sum_{k=1}^{m}\big(T'_{B_k^{(n)}} - T''_{B_k^{(n)}}\big) \qquad \text{for all } m,n \in \mathbb{N} \tag{6.13}$$

and the stopping time

$$\tau^{(n)} := \inf\big\{m \in \mathbb{N} : V_m^{(n)} = 1\big\}\,,$$

in analogy to the corresponding random variables in Subsection 6.1.2. Also recall that

$$T_{\tilde{a}_n,n} := \sum_{i=\tilde{a}_n+1}^{n} Y_i\,, \qquad \text{for all } n \in \mathbb{N},$$

from definition (6.1).

Lemma 6.5. *For every $n \in \mathbb{N}$ such that $n > \tilde{a}_n$ it follows that*

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{\tilde{a}_n,n}), \mathscr{L}(T_{\tilde{a}_n,n}+1)\big) \le \mathbb{P}\big[\tau^{(n)} > m\big] \quad \textit{for all } m \in \mathbb{Z}_+ \textit{ with } mL_n \le n. \tag{6.14}$$

Proof. Fix $n \in \mathbb{N}$ with $n > \tilde{a}_n$. If $m = 0$ we have $\mathbb{P}[\tau^{(n)} > m] = 1$, which yields (6.14). Let $m \ge 1$. The lemma is a direct consequence of the coupling inequality described by Lindvall (2002, Section I.2). Each process $\{T'_{B_m^{(n)}}\}_{m\in\mathbb{N}}$ and $\{T''_{B_m^{(n)}}\}_{m\in\mathbb{N}}$ consists of mutually independent random variables. Thus, if we define

$$T'''_{B_m^{(n)}} := \begin{cases} T''_{B_m^{(n)}} & \text{if } m \le \tau^{(n)}, \\ T'_{B_m^{(n)}} & \text{if } m > \tau^{(n)}, \end{cases}$$

it follows that

$$\mathscr{L}\Big(\sum_{k=1}^{m} T'''_{B_k^{(n)}}\Big) = \mathscr{L}\Big(\sum_{k=1}^{m} T''_{B_k^{(n)}}\Big) \qquad \text{for all } m \in \mathbb{N}.$$

In this way we have constructed a copy $\sum_{k=1}^{m} T'_{B_k^{(n)}}$ of the sum $\sum_{k=1}^{m} T_{B_k^{(n)}}$ and a copy $\sum_{k=1}^{m} T'''_{B_k^{(n)}}+1$ of the random variable $\sum_{k=1}^{m} T_{B_k^{(n)}}+1$ with the property that

$$\sum_{k=1}^{m} T'_{B_k^{(n)}} = \sum_{k=1}^{m} T'''_{B_k^{(n)}} + 1 \qquad \text{for all } m \geq \tau^{(n)}.$$

We conclude, invoking Lindvall (2002, inequality (2.8)), that for every $m \in \mathbb{N}$ the coupling inequality

$$d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum_{k=1}^{m} T_{B_k^{(n)}}\Big), \mathscr{L}\Big(\sum_{k=1}^{m} T_{B_k^{(n)}}+1\Big)\Big) \leq \mathbb{P}[\tau^{(n)} > m]$$

holds. If $mL_n \leq n$, we also have

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{\tilde{a}_n,n}), \mathscr{L}(T_{\tilde{a}_n,n}+1)\big) \leq d_{\mathrm{TV}}\Big(\mathscr{L}\Big(\sum_{k=1}^{m} T_{B_k^{(n)}}\Big), \mathscr{L}\Big(\sum_{k=1}^{m} T_{B_k^{(n)}}+1\Big)\Big),$$

which proves the lemma. □

In what follows, we give conditions on $\{Y_i\}_{i\in\mathbb{N}}$ in order to control the behaviour of the joint distributions of $T'_{B_k^{(n)}}$ and $T''_{B_k^{(n)}}$ in order to bound the probability $\mathbb{P}[\tau^{(n)} > M_n]$ in Lemma 6.5, M_n being the number of the highest block still contained entirely in $\{\tilde{a}_n+1, \ldots, n\}$.

6.3 Case: "$\mathrm{V_m}$ can reach level 1 directly"

As in the previous section, let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of mutually independent $\mathbb{Z}$-valued random variables. We make the following assumption in this section.

Assumption 6.6. There is an $N \in \mathbb{N}$ and there are constants $\psi_0 > 0$, $\psi_1 > 0$ such that there exist pairwise disjoint runs $R_1, R_2, \ldots$ of (ψ_0, ψ_1)-good indices in $\{N, N+1, \ldots\}$ which satisfy the following two properties:

(i) Denote the gap between two consecutive runs R_i and R_{i+1} by G_i. We require that

$$g := \sup_{i\in\mathbb{N}} |G_i| < \infty\,. \tag{6.15}$$

(ii) We also assume that length of each run is strictly larger than the length of any gap, that is

$$|R_i| > g \qquad \text{for all } i \in \mathbb{N}.$$

If $Y_i \sim i\mathrm{Be}(1/i)$ as in Example 6.3 then Assumption 6.6 is satisfied with $N = 2$, $\psi_0 = 1/2$, $\psi_1 = 1$ and $R_i := \{i\}$ for $i \geq 2$. Here, the gaps G_i have all length 0.

In view of Theorem 6.4, we consider the following lemma.

Lemma 6.7. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables which satisfies the uniformity condition $\mathsf{UC}(\mathfrak{r})$ for some positive integer sequence $\mathfrak{r}$ (cf. Definition 2.2), and which satisfies*

$$i\mathbb{E}Z_i = \theta_i + \mathrm{o}(1)\,, \tag{6.16}$$

where $\{\theta_i\}_{i\in\mathbb{N}}$ is the integer skeleton of the sinusoidal function $\Theta(t)$ from (6.4)*. Let $Y_i := iZ_i$ for all $i \in \mathbb{N}$. The sequence $\{Y_i\}_{i\in\mathbb{N}}$ satisfies Assumption 6.6, unless* (6.5a) *or* (6.5b) *holds true.*

Remark 6.8. We are only interested in the values of $\Theta(t)$ at integer arguments. At these values the variation at a frequency higher than 1/2, the Nyquist frequency, cannot be distinguished from variation at the corresponding frequency in $[0, 1/2]$. This is immediate from trigonometric arguments. Let $i \in \mathbb{N}$ and $1 \leq l \leq k$. If $f_l \geq 1$, we have $f_l^* := f_l - \lfloor f_l \rfloor \in [0, 1)$ and

$$\cos(2\pi f_l i - \varphi_l) = \cos(2\pi f_l^* i - \varphi_l)\,.$$

If $f_l \in [1/2, 1)$ we set $f_l^* := 1 - f_l \in (0, 1/2]$, and $\varphi_l^* := 2\pi - \varphi_l$ if $\varphi_l \in (0, 2\pi)$ and $\varphi_l^* := 0$ if $\varphi_l = 0$. Then we obtain

$$\cos(2\pi f_l i - \varphi_l) = \cos(-2\pi f_l i + \varphi_l + 2\pi(i-1)) = \cos(2\pi f_l^* i - \varphi_l^*)\,.$$

Therefore, there is a function $\Theta^*(t)$, defined as in (6.4), where all frequencies lie in the interval $[0, 1/2]$, which has the same integer skeleton $\{\theta_i\}_{i\in\mathbb{N}}$ as $\Theta(t)$.

Proof of Lemma 6.7. First, note that the sequence $\{Z_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{UQLC}(\theta, \mathfrak{r})$. This follows from Lemma 2.26, along with Lemma 2.22, Corollary 2.14 and Lemma 2.12. As in Remark 6.2 we conclude that

$$\lim_{i\to\infty} \mathbb{P}[Y_i = 0] = 1 \qquad \text{and} \qquad \mathbb{P}[Y_i = i] = \frac{\theta_i}{i} + \delta_i\,, \tag{6.17}$$

for some sequence $\delta_i = \mathrm{o}(1/i)$. Because of the convergence of $\mathbb{P}[Y_i = 0]$ to 1, we only have to examine the probabilities $\mathbb{P}[Y_i = i]$ in order to establish Assumption 6.6.

In view of Remark 6.8 there is a function $\Theta^*(t)$, defined as in (6.4), with frequencies $f_1^*, \ldots, f_k^* \in (0, 1/2]$ and integer skeleton $\{\theta_i\}_{i\in\mathbb{N}}$.

We distinguish different cases. If $\sum_{l=1}^{k} \lambda_l < \theta$, then $\{\theta_i\}_{i\in\mathbb{N}}$ is bounded away from 0; for N large enough, and $\psi_0, \psi_1 > 0$ chosen suitably small, every $i \geq N$ is a (ψ_0, ψ_1)-good index.

Now, let $\sum_{l=1}^{k} \lambda_l = \theta$. Assume that $f_{l_0}^* \in (0, 1/2)$ for some $1 \leq l_0 \leq k$. We consider

$$H_{l_0}(t) := \eta_{l_0} + \lambda_{l_0} \cos(2\pi f_{l_0}^* t - \varphi_{l_0}^*) \leq \Theta^*(t) \qquad \text{for all } t \in \mathbb{R}, \tag{6.18}$$

where

$$\eta_{l_0} := \theta - \sum_{\substack{l=1\\ l\neq l_0}}^{k} \lambda_l > 0\,.$$

The function $H_{l_0}(t)$ is a cosine function with period $p := 1/f_{l_0}^* > 2$. Because of $p > 2$, we may choose around each $t \in \mathbb{R}$ with $H_{l_0}(t) = 0$ a small interval $d_{t,p} := (t - \varepsilon_p, t + \varepsilon_p)$ in such a way that if $i \in d_{t,p}$, then $i + 1$ and $i + 2$ do not lie in any interval $d_{s,p}$ with $H_{l_0}(s) = 0$. It follows that there are runs of numbers $i \in \mathbb{N}$ where

$$H_{l_0}(i) \geq H_{l_0}(t_0 + \varepsilon_p) > 0 \qquad \text{for any fixed } t_0 \in \mathbb{R} \text{ with } H_{l_0}(t_0) = 0,$$

of length 2 ore more, with gaps of length at most 1 between them (which arise if i falls in an interval $d_{t,p}$). From (6.17) we conclude that there is an $N \in \mathbb{N}$ such that

$$i\mathbb{P}[Y_i = i] = \theta_i + i\delta_i \geq H_{l_0}(i) + i\delta_i \geq \frac{H_{l_0}(i)}{2} \geq \frac{H_{l_0}(t_0 + \varepsilon_p)}{2}$$

for all $i \geq N$ where i does not lie in any of the intervals $d_{t,p}$. Assumption 6.6 is satisfied.

Finally, assume that $\sum_{l=1}^{k} \lambda_l = \theta$ and $f_1^* = \cdots = f_k^* = 1/2$. If, for some $1 \leq l_0 \leq k$, $\varphi_{l_0}^* \in (0, \pi) \cup (\pi, 2\pi)$, then

$$\theta_i \geq H_{l_0}(i) \geq \min\{H_{l_0}(1), H_{l_0}(2)\} > 0 \qquad \text{for every } i \in \mathbb{N}.$$

Hence, for N large enough and $\psi_0, \psi_1 > 0$ suitably small, every $i \geq N$ is a (ψ_0, ψ_1)-good index.

If $\varphi_l^* \in \{0, \pi\}$ for all $1 \leq l \leq k$, but there are l_0 and l_1 with $\varphi_{l_0}^* = 0$ and $\varphi_{l_1}^* = \pi$, then

$$\begin{aligned} \theta_i &\geq H_{l_0}(i) = H_{l_0}(2) > 0 \qquad &&\text{for every even } i \in \mathbb{N}, \\ \theta_i &\geq H_{l_0}(i) = H_{l_0}(1) > 0 \qquad &&\text{for every odd } i \in \mathbb{N}. \end{aligned}$$

Once more, Assumption 6.6 is valid.

If either (6.5a) or (6.5b) hold, that is $\varphi_l^* = 0$ for all $1 \leq l \leq k$ or $\varphi_l^* = \pi$ for all $1 \leq l \leq k$, Assumption 6.6 is never satisfied. $\square$

6.3.1 The number of good pairs in a block

We assume that $\{Y_i\}_{i\in\mathbb{N}}$ satisfies Assumption 6.6 for $N \in \mathbb{N}$ and $\psi_0, \psi_1 > 0$; let $g \in \mathbb{Z}_+$ be the maximal length of a gap between two consecutive runs of (ψ_0, ψ_1)-good indices. As in Section 6.2, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of integers with $0 \le a_n < n$ for all $n \in \mathbb{N}$. Also recall the definition of $\tilde{a}_n := \tilde{a}_n(N)$ and $\bar{a}_n := \bar{a}_n(N)$ in (6.9).

For each $n \in \mathbb{N}$, we consider blocks $B_m^{(n)}$, $m \in \mathbb{N}$, of length L_n as in (6.10). Here we require that the block lengths L_n have a special form, in contrast to Section 6.2, where we imposed no restrictions on L_n. Indeed, taking the sequence $\{a_n\}_{n\in\mathbb{N}}$ into account, we generalize the definition of l_n in (6.8) to

$$l_n := l_n(\varepsilon) := \Big\lfloor \frac{\varepsilon}{\log 2} \log\Big(\frac{n}{\bar{a}_n}\Big) \Big\rfloor \qquad \text{for all } n \in \mathbb{N}, \tag{6.19}$$

where $0 < \varepsilon < 1$ is fixed. We then extend the block lengths L_n of (6.8) to

$$L_n := L_n(\varepsilon) := R\, 2^{l_n(\varepsilon)} \qquad \text{for all } n \in \mathbb{N}, \tag{6.20}$$

where

$$R := R(g) := 5g + 1\,, \tag{6.21}$$

recalling that g, defined in (6.15), is the length of the longest gap between two consecutive runs of good indices. The reason for this choice of R will become apparent in Lemma 6.9 below. Essentially, with R as in (6.21), there are enough pairs of (ψ_0, ψ_1)-good indices in order to establish a coupling as described in Subsection 6.1.2 despite gaps between runs of good indices. If every natural number $i \ge N$ is a (ψ_0, ψ_1)-good index, we have $g = 0$ and thus $L_n = 2^{l_n}$ as in (6.8). Note that

$$\frac{1}{2}\Big(\frac{n}{\bar{a}_n}\Big)^{\varepsilon} \le L_n(\varepsilon) \le R\Big(\frac{n}{\bar{a}_n}\Big)^{\varepsilon} \qquad \text{for all } n \in \mathbb{N}.$$

Lemma 6.9. *Let $n \in \mathbb{N}$ be such that $l_n \ge 2$ (or, equivalently, $L_n \ge 4R$). For every $m \in \mathbb{N}$ and for every $0 \le k < l_n$ there is a partition $\mathcal{P}_m^{(n)}(k)$ of block $B_m^{(n)}$ into pairwise disjoint sets, among which there are*

$$K_n \ge \frac{L_n}{4R} \tag{6.22}$$

(ψ_0, ψ_1)-good pairs (i, j) (as in Definition 6.1) satisfying

$$j - i = 2^k\,. \tag{6.23}$$

Remark 6.10. Note that the number K_n of (ψ_0, ψ_1)-good pairs in $\mathcal{P}_m^{(n)}(k)$ trivially satisfies $K_n \leq L_n/2$.

Example 6.11. Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of mutually independent $\mathbb{Z}$-valued random variables that satisfies Assumption 6.6, with the property that there is an $N \in \mathbb{N}$ and constants $\psi_0, \psi_1 > 0$ such that every integer $i \geq N$ is a (ψ_0, ψ_1)-good index. In this case there are no gaps; we have $g = 0$ and $R = 1$ (cf. (6.15) and (6.21)). Then, with n as in Lemma 6.9, for each $m \in \mathbb{N}$ and $0 \leq k < l_n$ there is a partition $\mathcal{P}_m^{(n)}(k)$ that contains

$$K_n = \frac{L_n}{2}$$

(ψ_0, ψ_1)-good pairs.

Proof of Lemma 6.9. Recall from (6.15) that $g \in \mathbb{Z}_+$ is the maximal length of a gap between two consecutive runs of (ψ_0, ψ_1)-good indices.

Now every run of (ψ_0, ψ_1)-good indices of length $2(g + 1)$, or more, is split into runs of (ψ_0, ψ_1)-good indices whose lengths are larger than g and smaller than $2(g + 1)$, with gaps of length 0 in between. Hence, the length of a run of (ψ_0, ψ_1)-good indices is at most $2g + 1$, and the length of two runs of (ψ_0, ψ_1)-good indices with a gap between them cannot exceed $5g + 2$. Then a sequence of consecutive natural numbers of length $R = 5g + 1$ then contains at least one entire run of (ψ_0, ψ_1)-good indices.

Because runs of (ψ_0, ψ_1)-good indices are longer than the gaps between them, we find for each $d \in \mathbb{N}$ a (ψ_0, ψ_1)-good index i in each run, and some other (ψ_0, ψ_1)-good index $j > i$, possibly in another run, such that $j - i = d$.

Fix an $n \in \mathbb{N}$ with $l_n \geq 2$. Let $0 \leq k < l_n$ and set $d := 2^k$. In each block $B_m^{(n)}$ we find $2^{c_n-2} = L_n/(4R)$ non-overlapping 2-sets $\{i, j\}$ of (ψ_0, ψ_1)-good indices satisfying $j - i = d = 2^k$ as follows.

Divide $B_m^{(n)}$ into 2^{l_n} small blocks of length $R = 5g + 1$. Consider the first of these small blocks. By construction, it contains at least one entire run of (ψ_0, ψ_1)-good indices. There we find a (ψ_0, ψ_1)-good index i_1, such that there is another (ψ_0, ψ_1)-good index j_1 with $j_1 - i_1 = 2^k$. Since $j_1 \leq R + 2^k \leq R2^{l_n} = L_n$, we have $j_1 \in B_m^{(n)}$. By choosing these two indices, we "use up" at most two of the small blocks in the *first half* of $B_m^{(n)}$. Indeed, we use up one block if i_1 and j_1 lie in the same small block, or if j_1 lies in the second half of $B_m^{(n)}$. We use up two blocks if i_1 and j_1 lie in two different small blocks in the first half of $B_m^{(n)}$.

Then we proceed to the next "remaining" small block in the first half of $B_m^{(n)}$. There, too, we find a (ψ_0, ψ_1)-good index, i_2 say, such that $j_2 - i_2 = 2^k$ for some (ψ_0, ψ_1)-good index $j_2 \in B_m^{(n)}$. Again, at most two small blocks in the first half of $B_m^{(n)}$ are used up.

If we proceed in this manner, at least half of all the small blocks in the first half of $B_m^{(n)}$ are available for picking the indices i_r. Since $L_n = R2^{l_n}$, there are at least 2^{l_n-2} runs of (ψ_0, ψ_1)-good indices in the first half of $B_m^{(n)}$ to choose the indices i_r from. Thus, we have non-overlapping pairs $\{i_r, j_r\} \subset B_m^{(n)}$ of (ψ_0, ψ_1)-good indices, with $j_r - i_r = 2^k$ for $1 \leq r \leq 2^{l_n-2} = L_n/(4R)$. □

6.3.2 Definition of the joint distributions

Again, let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of mutually independent $\mathbb{Z}$-valued random variables that satisfies Assumption 6.6 for $N \in \mathbb{N}$ and $\psi_0, \psi_1 > 0$. Fix $m \in \mathbb{N}$ and let (i, j) be a (ψ_0, ψ_1)-good pair in block $B_m^{(n)}$. It follows that

$$\begin{aligned} &\min\bigl\{\mathbb{P}[Y_i + Y_j = i], \mathbb{P}[Y_i + Y_j = j]\bigr\} \\ &\geq \min\bigl\{\mathbb{P}[Y_i = i, Y_j = 0], \mathbb{P}[Y_i = 0, Y_j = j]\bigr\} \geq \frac{\psi_0\psi_1}{j} \geq \frac{\psi_0\psi_1}{(w_n + m)L_n}\,. \end{aligned} \tag{6.24}$$

Now assume that $l_n \geq 2$ in order to satisfy the assumptions of Lemma 6.9. For each $m \in \mathbb{N}$ and $0 \leq k < l_n$ we choose a partition $\mathcal{P}_m^{(n)}(k)$ of block $B_m^{(n)}$ as in Lemma 6.9 with K_n (ψ_0, ψ_1)-good pairs

$$(i_{m,1}, i_{m,1} + 2^k), \ldots, (i_{m,K_n}, i_{m,K_n} + 2^k)\,, \tag{6.25}$$

where $i_{m,r} < i_{m,r+1}$ for $1 \leq r < K_n$. Considering $\mathcal{P}_m^{(n)}(k)$ and the pairs (6.25) as fixed, we set

$$S_{m,r}^{(n)}(k) := Y_{i_{m,r}} + Y_{i_{m,r}+2^k} \qquad \text{for all } 1 \leq r \leq K_n.$$

We also consider copies $S_{m,r}^{(n)}(k)'$ of $S_{m,r}^{(n)}(k)$ for each $m \in \mathbb{N}$, $0 \leq k < l_n$, and $1 \leq r < K_n$, all defined on a common probability space with $\{Y_i\}_{i\in\mathbb{N}}$. We assume that the random vectors

$$\bigl(S_{m,r}^{(n)}(k_m), S_{m,r}^{(n)}(k_m)'\bigr)\,, \qquad m \in \mathbb{N},\, 1 \leq r < K_n\,,$$

with $0 \leq k_m < l_n$ for each $m \in \mathbb{N}$, are mutually independent, and that they are also independent of

$$\bigl\{Y_j : j \notin \{i_{m,r}, i_{m,r} + 2^{k_m}\} \text{ for any } m \in \mathbb{N} \text{ and } 1 \leq r \leq K_n\bigr\}\,.$$

In view of (6.24) we can define joint distributions of $S^{(n)}_{m,r}(k)$ and $S^{(n)}_{m,r}(k)'$ by

$$
\begin{aligned}
\mathbb{P}\Big[\big(S^{(n)}_{m,r}(k), S^{(n)}_{m,r}(k)'\big) = (i_{m,r}, i_{m,r} + 2^k)\Big] &:= \frac{\psi_0\psi_1}{(w_n+m)L_n} \wedge \frac{1}{2} > 0\,,\\
\mathbb{P}\Big[\big(S^{(n)}_{m,r}(k), S^{(n)}_{m,r}(k)'\big) = (i_{m,r} + 2^k, i_{m,r})\Big] &:= \frac{\psi_0\psi_1}{(w_n+m)L_n} \wedge \frac{1}{2} > 0\,,\\
\mathbb{P}\Big[S^{(n)}_{m,r}(k) = S^{(n)}_{m,r}(k)'\Big] &:= 1 - 2\Big(\frac{\psi_0\psi_1}{(w_n+m)L_n} \wedge \frac{1}{2}\Big)\,.
\end{aligned}
\tag{6.26}
$$

It is immediate that

$$
S^{(n)}_{m,r}(k) - S^{(n)}_{m,r}(k)' \in \big\{-2^k, 0, 2^k\big\}\,,
$$

and that

$$
\begin{aligned}
\mathbb{P}\big[S^{(n)}_{m,r}(k) - S^{(n)}_{m,r}(k)' = -2^k \;\big|\; S^{(n)}_{m,r}(k) \neq S^{(n)}_{m,r}(k)'\big] &= \frac{1}{2}\,,\\
\mathbb{P}\big[S^{(n)}_{m,r}(k) - S^{(n)}_{m,r}(k)' = 2^k \;\big|\; S^{(n)}_{m,r}(k) \neq S^{(n)}_{m,r}(k)'\big] &= \frac{1}{2}\,.
\end{aligned}
$$

We introduce a stopping time

$$
\tau^{(n)}_m(k) := \inf\big\{1 \le r \le K_n : S^{(n)}_{m,r}(k) \neq S^{(n)}_{m,r}(k)'\big\}\,, \tag{6.27}
$$

which gives the index of the first of our K_n fixed pairs $(i_{m,r}, i_{m,r} + 2^k)$ where a S-jump, that is, an event of the form $S^{(n)}_{m,r}(k) \neq S^{(n)}_{m,r}(k)'$, occurs. If none of these events occur, we have $\tau^{(n)}_m(k) = \infty$. Let

$$
R^{(n)}_m := B^{(n)}_m \setminus \bigcup_{r=1}^{K_n} \{i_{m,r}, i_{m,r} + 2^k\}
$$

be an auxiliary subset of $B^{(n)}_m$ that collects the remaining indices in the block. We then introduce the random variables

$$
S^{(n)}_m(k) := \mathbf{1}\{k < l_n\}\Big(\sum_{r=1}^{K_n} S^{(n)}_{m,r}(k) + \sum_{i \in R^{(n)}_m} Y_i\Big) + \mathbf{1}\{k = l_n\} \sum_{i \in B^{(n)}_m} Y_i\,, \tag{6.28}
$$

$$
\begin{aligned}
S^{(n)}_m(k)' := \mathbf{1}\{k < l_n\}\Big(&\sum_{k=1}^{\tau^{(n)}_m(k)\wedge K_n} S^{(n)}_{m,r}(k)' + \sum_{k=\tau^{(n)}_m(k)+1}^{K_n} S^{(n)}_{m,r}(k) + \sum_{i \in R^{(n)}_m} Y_i\Big)\\
&+ \mathbf{1}\{k = l_n\} \sum_{i \in B^{(n)}_m} Y_i\,,
\end{aligned}
\tag{6.29}
$$

their difference
$$U_m^{(n)}(k) := S_m^{(n)}(k) - S_m^{(n)}(k)', \tag{6.30}$$
and the indicator
$$I_m^{(n)}(k) := \mathbf{1}\big\{U_m^{(n)}(k) \neq 0\big\}. \tag{6.31}$$

By construction, the random vectors
$$\big(S_m^{(n)}(k_m), S_m^{(n)}(k_m)'\big), \qquad m \in \mathbb{N},$$
where $1 \leq k_m \leq l_n$, are independent. Thus, the sequences $\{U_m^{(n)}(k_m)\}_{m\in\mathbb{N}}$ and $\{I_m^{(n)}(k_m)\}_{m\in\mathbb{N}}$ consist of independent random variables as well.

Lemma 6.12. *(i) The distributions of $S_m^{(n)}(k)$ and $S_m^{(n)}(k)'$ do not depend on the underlying partition $\mathcal{P}_m^{(n)}(k)$ of $B_m^{(n)}$. More precisely,*
$$\mathscr{L}\big(S_m^{(n)}(k)\big) = \mathscr{L}\big(S_m^{(n)}(k)'\big) = \mathscr{L}\big(T_{B_m^{(n)}}\big) \qquad \text{for all } 0 \leq k \leq l_n,$$
with $T_{B_m^{(n)}}$ defined in (6.12).

(ii) Let $0 \leq k < l_n$. The probability of a U-jump in block $B_m^{(n)}$, that is an event of the form $U_m^{(n)}(k) \neq 0$, is strictly positive and does not depend on the underlying partition $\mathcal{P}_m^{(n)}(k)$. Indeed, it is equal to
$$p_m^{(n)} := \mathbb{P}\big[I_m^{(n)}(k) = 1\big] = 1 - \left(1 - 2\Big(\frac{\psi_0\psi_1}{(w_n+m)L_n} \wedge \frac{1}{2}\Big)\right)^{K_n}. \tag{6.32}$$
What is more,
$$\tilde{p}_m^{(n)} - (\tilde{p}_m^{(n)})^2 \leq p_m^{(n)} \leq \tilde{p}_m^{(n)},$$
where
$$\frac{\psi_0\psi_1}{2R(w_n+m)} \wedge K_n \leq \tilde{p}_m^{(n)} := 2K_n\Big(\frac{\psi_0\psi_1}{(w_n+m)L_n} \wedge \frac{1}{2}\Big) \leq \frac{\psi_0\psi_1}{w_n+m}. \tag{6.33}$$
If $k = l_n$, we have $I_m^{(n)}(k) = 1$.

(iii) For every $0 \leq k < l_n$, we have $U_m^{(n)}(k) \in \{-2^k, 0, 2^k\}$ and
$$\mathbb{P}\big[U_m^{(n)}(k) = -2^k \mid I_m^{(n)}(k) = 1\big] = \frac{1}{2}$$
$$\mathbb{P}\big[U_m^{(n)}(k) = 2^k \mid I_m^{(n)}(k) = 1\big] = \frac{1}{2}.$$
If $k = l_n$, it follows that $U_m^{(n)}(k) = 0$.

Proof. (i) If $k < l_n$, this follows because $(S^{(n)}_{m,r}(k), S^{(n)}_{m,r}(k)')$, $1 \leq r \leq K_n$, are mutually independent, and the distribution of both $S^{(n)}_{m,r}(k)$ and $S^{(n)}_{m,r}(k)'$ is the same as the distribution of $Y_{i_{m,r}} + Y_{i_{m,r}+2^k}$ for every $1 \leq r \leq K_n$. The case $k = l_n$ is trivial.

(ii) If $k < l_n$, the independence of $(S^{(n)}_{m,r}(k), S^{(n)}_{m,r}(k)')$, $1 \leq r \leq K_n$, and (6.26) entail that

$$\begin{aligned}\mathbb{P}\big[I^{(n)}_m(k) = 1\big] &= 1 - \prod_{r=1}^{K_n} \mathbb{P}\big[S^{(n)}_{m,r}(k) = S^{(n)}_{m,r}(k)'\big] \\ &= 1 - \left(1 - 2\Big(\frac{\psi_0\psi_1}{(w_n+m)L_n} \wedge \frac{1}{2}\Big)\right)^{K_n}.\end{aligned}$$

The upper bound of $p^{(n)}_m$ is immediate from the Bernoulli inequality. The lower bound follows from the second Bonferroni inequality. The bounds on $\tilde{p}^{(n)}_m$ are immediate from Lemma 6.9 and Remark 6.10. If $k = l_n$, we have $S^{(n)}_m(k) = S^{(n)}_m(k)'$.

(iii) These assertions are a direct consequence of the joint distribution defined in (6.26) and the definition of the stopping time $\tau^{(n)}_m(k)$. □

Remark 6.13. Let $0 \leq k_m < l_n$ for all $m \in \mathbb{N}$. Then, because of Lemma 6.12 (ii), the sequence $\{I^{(n)}_m(k_m)\}_{m\in\mathbb{N}}$ consists of independent Bernoulli random variables whose distributions do not depend on k_m. We therefore can represent $\{I^{(n)}_m(k_m)\}_{m\in\mathbb{N}}$ by a sequence $\{I^{(n)}_m\}_{m\in\mathbb{N}}$ of independent Bernoulli random variables,

$$I^{(n)}_m \sim \mathrm{Be}(p^{(n)}_m) \qquad \text{for all } m \in \mathbb{N},$$

$p^{(n)}_m$ defined in (6.32).

6.3.3 Jump counts

As in the previous subsection, let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of mutually independent $\mathbb{Z}$-valued random variables that satisfies Assumption 6.6 for $N \in \mathbb{N}$ and $\psi_0, \psi_1 > 0$. Assume that $l_n \geq 2$, and fix a partition $\mathcal{P}^{(n)}_m(k)$ as in Lemma 6.9 for every $m \in \mathbb{N}$ and $0 \leq k < l_n$. We assume that the random variables $S^{(n)}_m(k)$, $S^{(n)}_m(k)'$, $U^{(n)}_m(k)$ and $I^{(n)}_m(k)$, defined in (6.28), (6.29), (6.30) and (6.31), respectively, are based on these fixed partitions.

In this subsection, we replace the fixed parameter k by a random variable that counts the number of U-jumps (cf. Lemma 6.12 (ii)) that have occurred in the preceding blocks. More formally, we define

$$J^{(n)}_0 := 0 \qquad \text{and} \qquad J^{(n)}_m := J^{(n)}_{m-1} + I^{(n)}_m(J^{(n)}_{m-1}) \quad \text{for all } m \in \mathbb{N}.$$

The recursive definition yields

$$J_{m-1}^{(n)} \le J_m^{(n)} \qquad \text{for all } m \in \mathbb{N}$$

What is more, $(S_m^{(n)}(k), S_m^{(n)}(k)')$ and $J_{m-1}^{(n)}$ are independent for each $m \in \mathbb{N}$ and $0 \le k \le l_n$. So are $I_m^{(n)}(k)$ and $J_{m-1}^{(n)}$, and Lemma 6.12 (ii) yields

$$J_m^{(n)} \le l_n \qquad \text{for all } m \in \mathbb{N}.$$

We introduce the random vector

$$\bar{I}_m^{(n)} := \big(I_1^{(n)}(J_0^{(n)}),\, I_2^{(n)}(J_1^{(n)}),\, \ldots,\, I_m^{(n)}(J_{m-1}^{(n)})\big)\,, \tag{6.34}$$

which describes the jump structure up to block $B_m^{(n)}$. It is useful in the proof of the next lemma.

Lemma 6.14. *Let $n \in \mathbb{N}$ such that $l_n \ge 2$, and assume that*

$$\frac{\psi_0\psi_1}{(w_n+1)L_n} \le \frac{1}{2}\,. \tag{6.35}$$

Then we have for every $x > 0$

$$\mathbb{P}\big[J_m^{(n)} < l_n\big] \le \exp\big((1-\mathrm{e}^{-x})\psi_0^2\psi_1^2\pi^2/6\big)\mathrm{e}^{xl_n}\Big(\frac{w_n+1}{m+w_n+1}\Big)^{2\psi_0\psi_1(1-\mathrm{e}^{-x})K_n/L_n}\,,$$

with $K_n/L_n \ge 1/(4R)$.

Proof. We introduce the auxiliary set

$$E := \Big\{\bar{e} = (e_1, \ldots, e_m) \in \{0,1\}^m : \textstyle\sum_{j=1}^m e_j < l_n\Big\}\,,$$

which is non-empty under our assumption that $l_n \ge 2$. From the independence of the jump indicators $\{I_m^{(n)}(k_m)\}_{m\in\mathbb{N}}$, where $0 \le k_m < l_n$, it follows that

$$\begin{aligned}
\mathbb{P}\big[J_m^{(n)} < l_n\big] &= \sum_{\bar{e}\in E} \mathbb{P}\big[\bar{I}_m^{(n)} = \bar{e}\big] \\
&= \sum_{\bar{e}\in E} \mathbb{P}\Big[\big(I_1^{(n)}(0), \ldots, I_m^{(n)}(\textstyle\sum_{j=m'+1}^{m-1} e_j)\big) = \bar{e}\Big] \\
&= \sum_{\bar{e}\in E} \mathbb{P}\Big[\big(I_1^{(n)}, \ldots, I_m^{(n)}\big) = \bar{e}\Big] \\
&= \mathbb{P}\Big[\textstyle\sum_{r=1}^m I_r^{(n)} < l_n\Big]\,.
\end{aligned}$$

We invoke Chernov bounds (Ross, 1996, Section 1.7) to obtain

$$\mathbb{P}\Big[\sum\nolimits_{r=1}^{m} I_r^{(n)} < l_n\Big] \le \mathrm{e}^{x l_n}\mathbb{E}\Big\{\exp\Big(-x\sum\nolimits_{r=1}^{m} I_r^{(n)}\Big)\Big\}$$

for arbitrary $x > 0$. Since $I_1^{(n)}, \dots, I_m^{(n)}$ are mutually independent, the moment generating function of the sum of these random variables can be estimated by

$$\begin{aligned}\mathbb{E}\Big\{\exp\Big(-x\sum\nolimits_{r=1}^{m} I_r^{(n)}\Big)\Big\} &= \prod_{r=1}^{m}\Big(1-\mathbb{P}\big[I_r^{(n)}=1\big]\big(1-\mathrm{e}^{-x}\big)\Big)\\ &\le \exp\Big(-(1-\mathrm{e}^{-x})\sum\nolimits_{r=1}^{m}\mathbb{P}\big[I_r^{(n)}=1\big]\Big)\,,\end{aligned}$$

which can be bounded further, using Lemma 6.12 (ii) with (6.35), and (A.3), by

$$\begin{aligned}&\exp\Big(-(1-\mathrm{e}^{-x})\sum\nolimits_{r=1}^{m}\tilde{p}_r^{(n)} + (1-\mathrm{e}^{-x})\sum\nolimits_{r=1}^{m}(\tilde{p}_r^{(n)})^2\Big)\\ &\quad\le \exp\big((1-\mathrm{e}^{-x})\psi_0^2\psi_1^2\pi^2/6\big)\left(\frac{w_n+1}{m+w_n+1}\right)^{2\psi_0\psi_1(1-\mathrm{e}^{-x})K_n/L_n}.\end{aligned}$$

This proves the lemma. □

6.3.4 Total variation bounds

We return to the coupling inequality from Section 6.2, but we keep the assumptions from the beginning of Subsection 6.3.3 in force. The next lemma allows us to choose

$$T'_{B_m^{(n)}} := S_m^{(n)}(J_{m-1}^{(n)}) \quad\text{and}\quad T''_{B_m^{(n)}} := S_m^{(n)}(J_{m-1}^{(n)})'\,, \qquad m\in\mathbb{N}\,,$$

as copies of the process $\{T_{B_m^{(n)}}\}_{m\in\mathbb{N}}$ in order to define $\{V_m\}_{m\in\mathbb{N}}$ as in (6.13).

Lemma 6.15. *The sequences*

$$\big\{S_m^{(n)}(J_{m-1}^{(n)})\big\}_{m\in\mathbb{N}} \qquad\textit{and}\qquad \big\{S_m^{(n)}(J_{m-1}^{(n)})'\big\}_{m\in\mathbb{N}}$$

consist of independent random variables. Moreover,

$$\mathscr{L}\big(S_m^{(n)}(J_{m-1}^{(n)})\big) = \mathscr{L}\big(S_m^{(n)}(J_{m-1}^{(n)})'\big) = \mathscr{L}\big(T_{B_m^{(n)}}\big) \qquad \textit{for all } m\in\mathbb{N}. \tag{6.36}$$

Proof. Equation (6.36) follows from the independence of $(S_m^{(n)}(k), S_m^{(n)}(k)')$ and $J_{m-1}^{(n)}$, and from Lemma 6.12 (i).

We prove the independence of $\{S_m^{(n)}(J_{m-1}^{(n)})'\}_{m\in\mathbb{N}}$. It is enough to show that

$$\bar{S}_{m-1}^{(n)} := \big(S_1^{(n)}(J_0^{(n)})', \dots, S_{m-1}^{(n)}(J_{m-2}^{(n)})'\big) \quad \text{and} \quad S_m^{(n)}(J_{m-1}^{(n)})'$$

are independent for every $m \geq 2$. But this follows, because $\{\bar{S}_{m-1}^{(n)}, J_{m-1}^{(n)}\}$ is independent of $S_m^{(n)}(k)'$ for every $0 \leq k \leq l_n$, and because of (6.36). $\square$

Recall the definition of $l_n := l_n(\varepsilon)$ in (6.19), $L_n := L_n(\varepsilon) := R2^{l_n(\varepsilon)}$ in (6.20), and of $w_n := w_n(\varepsilon) := \lceil \tilde{a}_n / L_n(\varepsilon) \rceil$ in (6.11). We set

$$M_n := M_n(\varepsilon) := \Big\lfloor \frac{n}{L_n(\varepsilon)} - w_n(\varepsilon) \Big\rfloor \qquad \text{for all } n \in \mathbb{N}.$$

We have $(w_n + M_n)L_n \leq n$, and, if also $M_n \geq 1$, M_n is the number of the highest block still contained entirely in $\{\tilde{a}_n + 1, \dots, n\}$. From Lemma 6.5 we conclude that

$$\begin{aligned} & d_{\mathrm{TV}}\big(\mathscr{L}(T_{\tilde{a}_n,n}), \mathscr{L}(T_{\tilde{a}_n,n}+1)\big) \\ &\quad \leq \mathbb{P}[\tau^{(n)} > M_n] \\ &\quad = \mathbb{P}[V_m \neq 1 \text{ for } 1 \leq m \leq M_n] \\ &\quad \leq \mathbb{P}[J_{M_n}^{(n)} < l_n] + \mathbb{P}[J_{M_n}^{(n)} = l_n,\, V_m \neq 1 \text{ for } 1 \leq m \leq M_n]\,. \end{aligned} \tag{6.37}$$

Lemma 6.16. *Let* $0 < \varepsilon < 1$. *Let* $n \in \mathbb{N}$ *such that* $l_n(\varepsilon) \geq 2$ *and such that*

$$\frac{\psi_0\psi_1}{(w_n+1)L_n} \leq \frac{1}{2}\,.$$

The we have

$$\mathbb{P}\big[J_{M_n}^{(n)} < l_n\big] \leq \rho(\psi_0,\psi_1,R)\Big(\frac{\bar{a}_n}{n}\Big)^{\psi_0\psi_1/(4R)-\varepsilon(1+\psi_0\psi_1/(4R))},$$

where $\rho(\psi_0,\psi_1,R) := \exp(\psi_0^2\psi_1^2\pi^2/12)(3R)^{\psi_0\psi_1/2}$.

Proof. Fix $0 < \varepsilon < 1$. Choosing $x := \log 2$ and $m := M_n$ in Lemma 6.14 it follows that

$$\begin{aligned} \mathbb{P}\big[J_{M_n}^{(n)} < l_n\big] &\leq \exp\big(\psi_0^2\psi_1^2\pi^2/12\big)2^{l_n}\Big(\frac{w_n+1}{m+w_n+1}\Big)^{\psi_0\psi_1 K_n/L_n} \\ &\leq \exp\big(\psi_0^2\psi_1^2\pi^2/12\big)\Big(\frac{\bar{a}_n}{n}\Big)^{\varepsilon}\Big(3R\frac{\bar{a}_n \vee (n/\bar{a}_n)^{\varepsilon}}{n}\Big)^{\psi_0\psi_1 K_n/L_n}. \end{aligned}$$

We distinguish the cases $\bar{a}_n \leq (n/\bar{a}_n)^{\varepsilon}$ and $\bar{a}_n > (n/\bar{a}_n)^{\varepsilon}$, and we invoke Lemma 6.9 and Remark 6.10. This proves the lemma. $\square$

Lemma 6.17. *Let $0 < \varepsilon < 1$. Let $n \in \mathbb{N}$ such that $l_n(\varepsilon) \geq 2$ and $M_n(\varepsilon) \geq 1$. Then it follows that*

$$\mathbb{P}[J^{(n)}_{M_n} = l_n,\, V_m \neq 1 \text{ for } 1 \leq m \leq M_n] \leq \frac{1}{2^{l_n}} \leq 2\Big(\frac{\bar{a}_n}{n}\Big)^{\varepsilon}.$$

Proof. We define the auxiliary set

$$E := \Big\{\bar{e} = (e_1, \ldots, e_{M_n}) \in \{0,1\}^{M_n} : \sum\nolimits_{i=1}^{M_n} e_i = l_n\Big\} \neq \emptyset\,,$$

and we write $k_r := \sum_{i=1}^{r} e_i$, for $1 \leq r \leq M_n$, and $k_0 := 0$. Also recall the definition of $\bar{I}^{(n)}_{M_n}$ in (6.34). Then

$$\begin{aligned}
&\mathbb{P}[J^{(n)}_{M_n} = l_n,\, V_m \neq 1 \text{ for } 1 \leq m \leq M_n] \\
&\quad = \sum_{\bar{e}\in E} \mathbb{P}[\bar{I}^{(n)}_{M_n} = \bar{e},\, V_m \neq 1 \text{ for } 1 \leq m \leq M_n] \\
&\quad = \sum_{\bar{e}\in E} \prod_{r:\, e_r=0} \mathbb{P}\big[I^{(n)}_r(k_{r-1}) = 0\big] \prod_{r:\, e_r=1} \mathbb{P}\big[I^{(n)}_r(k_{r-1}) = 1, U^{(n)}_r(k_{r-1}) = -2^{k_{r-1}}\big] \\
&\quad = \sum_{\bar{e}\in E} \prod_{r:\, e_r=0} \mathbb{P}\big[I^{(n)}_r(k_{r-1}) = 0\big] \prod_{r:\, e_r=1} \frac{\mathbb{P}\big[I^{(n)}_r(k_{r-1}) = 1\big]}{2} \\
&\quad = \frac{\mathbb{P}[J^{(n)}_{M_n} = l_n]}{2^{l_n}}\,,
\end{aligned}$$

where the third equality is from Lemma 6.12 (iii). The lemma now follows from (6.19). ☐

We collect the results of this section in the following theorem.

Theorem 6.18. *Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}$ -valued random variables that satisfies Assumption 6.6 for $N \in \mathbb{N}$ and $\psi_0, \psi_1 > 0$; let $g \in \mathbb{Z}_+$ be the maximal length of a gap between two consecutive runs of (ψ_0, ψ_1)-good indices, and $R := 5g + 1$. Let*

$$\varepsilon := \frac{\psi_0\psi_1/(4R)}{2 + \psi_0\psi_1/(4R)}\,.$$

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of integers with $0 \leq a_n < n$ for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ such that $l_n(\varepsilon) \geq 2$, $M_n(\varepsilon) \geq 1$ and such that

$$\frac{\psi_0\psi_1}{(w_n + 1)L_n(\varepsilon)} \leq \frac{1}{2}$$

it follows that

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{a_n,n}),\mathscr{L}(T_{a_n,n}+1)\big)\le(\rho(\psi_0,\psi_1,R)+2)\Big(\frac{\bar{a}_n}{n}\Big)^{\psi_0\psi_1/(\psi_0\psi_1+8R)},$$

with $\rho(\psi_0,\psi_1,R)$ as in Lemma 6.16. In particular, if $a_n=\mathrm{o}(n)$, the total variation distance between $\mathscr{L}(T_{a_n,n})$ and $\mathscr{L}(T_{a_n,n}+1)$ converges to 0.

Proof. Because $a_n\le\tilde{a}_n$, we have

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{a_n,n}),\mathscr{L}(T_{a_n,n}+1)\big)\le d_{\mathrm{TV}}\big(\mathscr{L}(T_{\tilde{a}_n,n}),\mathscr{L}(T_{\tilde{a}_n,n}+1)\big).$$

Now invoke (6.37), Lemma 6.16 and Lemma 6.17, and note that our choice of ε gives rise to a maximal exponent. □

6.4 Case: Using only odd indices

Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of mutually independent $\mathbb{Z}$-valued random variables. In this section, we make the following simple assumption.

Assumption 6.19. There is an $N\in\mathbb{N}$ and that there are constants $\psi_0>0$, $\psi_1>0$ such that each *odd* number $i\ge N$ is a (ψ_0,ψ_1)-good index.

The following lemma completes Lemma 6.7 in the sense that both lemmas together cover the assumption in Theorem 6.4.

Lemma 6.20. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables which satisfies the uniformity condition $\mathsf{UC}(\mathfrak{r})$ for some positive integer sequence $\mathfrak{r}$ (cf. Definition 2.2), and which satisfies*

$$i\mathbb{E}Z_i=\theta_i+\mathrm{o}(1),$$

where $\{\theta_i\}_{i\in\mathbb{N}}$ is the integer skeleton of the sinusoidal function $\Theta(t)$ from (6.4) *such that* (6.5a) *holds. Let $Y_i:=iZ_i$ for all $i\in\mathbb{N}$. The sequence $\{Y_i\}_{i\in\mathbb{N}}$ satisfies Assumption 6.19.*

Proof. As in the proof of Lemma 6.7, note that $\{Z_i\}_{i\in\mathbb{N}}$ satisfies (6.17). Under our assumption (6.5a) on $\Theta(t)$, we have $\theta_i=0$ for each even $i\in\mathbb{N}$, and, for some $\theta>0$, $\theta_i=\eta$ for each odd $i\in\mathbb{N}$. Assumption 6.19 is clearly satisfied. □

6.4.1 Blocks, good pairs

Assume that $\{Y_i\}_{i\in\mathbb{N}}$ satisfies Assumption 6.19 for $N \in \mathbb{N}$ and $\psi_0, \psi_1 > 0$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of integers with $0 \le a_n < n$ for all $n \in \mathbb{N}$. We define $\tilde{a}_n := \tilde{a}_n(N) := a_n \vee (N-1) \in \mathbb{Z}_+$ and $\bar{a}_n := \bar{a}_n(N) := \tilde{a}_n(N) \vee 1 \in \mathbb{N}$ as in (6.9).

We consider blocks similar to (6.10). Let

$$B_1^{(n)} := \{\tilde{a}_n + 1, \ldots, L_n\} \qquad \text{for all } n \in \mathbb{N}, \tag{6.38}$$

$$B_m^{(n)} := \{(m-1)L_n + 1, \ldots, mL_n\} \qquad \text{for all } n \in \mathbb{N} \text{ and } m \ge 2, \tag{6.39}$$

The block lengths L_n are specified as follows. Let

$$l_n := l_n(\varepsilon) := \left\lfloor \varepsilon \frac{\log(n/\bar{a}_n)}{\log 2} + \frac{\log \bar{a}_n}{\log 2} \right\rfloor \qquad \text{for all } n \in \mathbb{N}, \tag{6.40}$$

where $0 < \varepsilon < 1$ is fixed, and let

$$L_n := L_n(\varepsilon) := 2^{l_n(\varepsilon)} \qquad \text{for all } n \in \mathbb{N}. \tag{6.41}$$

It follows that

$$\frac{\bar{a}_n}{2}\left(\frac{n}{\bar{a}_n}\right)^{\varepsilon} \le L_n(\varepsilon) \le \bar{a}_n\left(\frac{n}{\bar{a}_n}\right)^{\varepsilon} \qquad \text{for all } n \in \mathbb{N}. \tag{6.42}$$

Assume that $l_n \ge 3$, or equivalently $L_n \ge 8$, and let $\delta \le L_n/2 = 2^{l_n - 1}$ be an even natural number. If $L_n \ge N-1$, every block $B_m^{(n)}$ with $m \ge 2$ lies in $\{N, N+1, \ldots\}$, and for any such m we find

$$K_n := \frac{L_n}{8} = 2^{l_n - 3} \in \mathbb{N} \tag{6.43}$$

pairwise disjoint subsets $\{i, j\}$ in $B_m^{(n)}$, where (i,j) is a (ψ_0, ψ_1)-good pair whose positive difference is $j - i = \delta$.

For our purposes, we only need to consider special cases of δ, constructed as follows. Let

$$h_n := h_n(\varepsilon) := \left\lfloor \frac{\varepsilon}{2} \frac{\log(n/\bar{a}_n)}{\log 2} + \frac{\log \bar{a}_n}{\log 2} \right\rfloor \qquad \text{for all } n \in \mathbb{N}, \tag{6.44}$$

and let

$$H_n := H_n(\varepsilon) := 2^{h_n(\varepsilon) - 1} \qquad \text{for all } n \in \mathbb{N}. \tag{6.45}$$

Note that

$$\frac{\bar{a}_n}{4}\left(\frac{n}{\bar{a}_n}\right)^{\varepsilon/2} \le H_n(\varepsilon) \le \frac{\bar{a}_n}{2}\left(\frac{n}{\bar{a}_n}\right)^{\varepsilon/2} \qquad \text{for all } n \in \mathbb{N}. \tag{6.46}$$

We also set

$$g_n := g_n(\varepsilon) := l_n(\varepsilon) - h_n(\varepsilon) \geq \frac{\varepsilon}{2} \frac{\log(n/\bar{a}_n)}{\log 2} - 1 \qquad \text{for all } n \in \mathbb{N}. \tag{6.47}$$

We are interested in δ of the form

$$\delta = (2d)2^k \qquad \text{for } 1 \leq d \leq H_n \text{ and } 0 \leq k < g_n$$

Note that for these choices of d and k we have, as required above, $\delta = (2d)2^k \leq 2^{l_n - 1} = L_n/2$.

In this context we also use the notation $A_+^{(n)} := \{a_n^*, \ldots, 2H_n - 1\} \subset 2\mathbb{N} - 1$, where a_n^* denotes the smallest odd number larger than or equal to $\tilde{a}_n + 1$. Moreover, we set $A_-^{(n)} := -A_+^{(n)}$ and $A^{(n)} := A_-^{(n)} \cup \{0\} \cup A_+^{(n)}$.

6.4.2 Definition of the joint distributions: The first block

We keep up the assumptions and notations of the previous subsection. For any odd index $i \geq N$, any of which is (ψ_0, ψ_1)-good by Assumption 6.19, we have

$$\min\big\{\mathbb{P}[Y_i = 0], \mathbb{P}[Y_i = i]\big\} \geq \frac{\psi_0 \wedge \psi_1}{i}\,. \tag{6.48}$$

Therefore, we can define a copy of the process $\{Y_i\}_{i \in \mathbb{N}}$ on a common probability space, $\{Y_i'\}_{i \in \mathbb{N}}$ say, with joint distributions for odd indices $i \geq N$ given by

$$\begin{aligned}
\mathbb{P}\big[(Y_i, Y_i') = (i, 0)\big] := \mathbb{P}\big[(Y_i, Y_i') = (0, i)\big] &:= \frac{\psi_0 \wedge \psi_1}{i} \wedge \frac{1}{2}\,, \\
\mathbb{P}\big[Y_i = Y_i'\big] &:= 1 - 2\Big(\frac{\psi_0 \wedge \psi_1}{i} \wedge \frac{1}{2}\Big)\,,
\end{aligned} \tag{6.49}$$

and where $\{(Y_i, Y_i')\}_{i \geq N}$ are mutually independent pairs.

Assume that $\tilde{a}_n < 2H_n - 1 \leq L_n$ holds true. In this case $A_+^{(n)} \subset B_1^{(n)}$ and $A_+^{(n)} \neq \emptyset$. Let

$$\tau_1^{(n)} := \inf\big\{j \in A_+^{(n)} : Y_j \neq Y_j'\big\} \tag{6.50}$$

be the odd number in $A_+^{(n)}$ at which a Y-jump, i. e. an event of the form $\{Y_j \neq Y_j'\}$, occurs for the first time. We set $\tau_1^{(n)} := \infty$, if $Y_j = Y_j'$ for each $j \in A_+^{(n)}$. Now, let

$$T'_{B_1^{(n)}} := \sum_{i=\tilde{a}_n+1}^{L_n} Y_i \quad \text{and} \quad T''_{B_1^{(n)}} := \sum_{j=a_n^*}^{\tau_1^{(n)} \wedge (2H_n - 1)} Y_j' + \sum_{j=\tau_1^{(n)}+1}^{2H_n - 1} Y_j' + \sum_{i \in R_1^{(n)}} Y_i'\,, \tag{6.51}$$

where $R_1^{(n)} := \{\tilde{a}_n + 1, \ldots, L_n\} \setminus A_+^{(n)}$. We also define the difference

$$U_1^{(n)} := T'_{B_1^{(n)}} - T''_{B_1^{(n)}}\,. \tag{6.52}$$

Lemma 6.21. *(i) Let $(\psi_0 \wedge \psi_1)/N \le 1/2$. Let $0 < \varepsilon < 1$ and let $H_n = H_n(\varepsilon)$ be defined as in* (6.45)*. The probability that no Y-jump occurs in block $B_1^{(n)}$ is*

$$\mathbb{P}[\tau_1^{(n)} = \infty] \le 3^{\psi_0\wedge\psi_1}\Big(\frac{\bar{a}_n}{H_n}\Big)^{\psi_0\wedge\psi_1} \le 12^{\psi_0\wedge\psi_1}\Big(\frac{\bar{a}_n}{n}\Big)^{\varepsilon\psi_0\wedge\psi_1/2}.$$

(ii) Both $T_{B_1^{(n)}}$ and $T'_{B_1^{(n)}}$ are identically $\mathscr{L}(\sum_{i=\tilde{a}_n+1}^{L_n} Y_i)$-distributed.

(iii) The difference $U_1^{(n)}$ is concentrated on the set $A^{(n)}$, and for each $j \in A_+^{(n)}$ we have

$$2\,\mathbb{P}\big[U_1^{(n)} = j,\, \tau_1^{(n)} = j\big] = 2\,\mathbb{P}\big[U_1^{(n)} = -j,\, \tau_1^{(n)} = j\big] = \mathbb{P}[\tau_1^{(n)} = j]\,.$$

Proof. To prove (i) we invoke (6.49) and the independence of the random vectors (Y_j, Y'_j), $j \in A_+^{(n)}$, to obtain

$$\begin{aligned}
\mathbb{P}[\tau_1^{(n)} = \infty] &= \prod_{j\in A_+^{(n)}} \mathbb{P}\big[Y_j = Y'_j\big] = \prod_{j\in A_+^{(n)}} \left(1 - 2\Big(\frac{\psi_0\wedge\psi_1}{j}\Big)\right) \\
&\le \exp\left(-2\psi_0\wedge\psi_1 \sum_{j=a_n^*,\, j\,\mathrm{odd}}^{2H_n-1} \frac{1}{j}\right) \le \exp\left(-\psi_0\wedge\psi_1 \sum_{j=(a_n^*+1)/2}^{H_n} \frac{1}{j}\right) \\
&\le \left(\frac{(a_n^*+1)/2+1}{H_n+1}\right)^{\psi_0\wedge\psi_1}.
\end{aligned}$$

We have $a_n^* + 1 \le 4\bar{a}_n$ for all $n \in \mathbb{N}$. This yields the first inequality of (i). For the second inequality, we then use that $H_n = 2^{h_n-1}$ and the definition of h_n in (6.44).

Part (ii) is immediate, and the assertions in (iii) are a direct consequence of the symmetry of the coupling. □

6.4.3 Definition of the joint distributions: The higher blocks

We assume that $K_n = 2^{l_n-3} \ge 1$, $H_n = 2^{h_n-1} \ge 1$ and $g_n = l_n - h_n \ge 1$. Throughout this subsection we consider a fixed block $B_m^{(n)}$ with $m \ge 2$. We also assume that $L_n \ge N - 1$. In this case, recalling that $m \ge 2$, every odd $i \in B_m^{(n)}$ is a (ψ_0, ψ_1)-good index.

For each pair (d,k), where $1 \le d \le H_n$ and $0 \le k < g_n$, we fix K_n (ψ_0, ψ_1)-good pairs in $B_m^{(n)}$ of the form

$$\big(i_{m,1}, i_{m,1} + (2d)2^k\big), \ldots, \big(i_{m,K_n}, i_{m,K_n} + (2d)2^k\big),$$

where $i_{m,1} < \cdots < i_{m,K_n}$, and introduce the auxiliary set

$$R_m^{(n)} := R_m^{(n)}(d,k) := B_m^{(n)} \setminus \bigcup_{r=1}^{K_n} \big\{ i_{m,r}, i_{m,r} + (2d)2^k \big\},$$

which collects the remaining indices in $B_m^{(n)}$. Considering these pairs fixed for each (d,k), we set

$$S_{m,r}^{(n)}(d,k) := Y_{i_{m,r}} + Y_{i_{m,r}+(2d)2^k} \qquad \text{for all } 1 \le r \le K_n.$$

We also consider copies $S_{m,r}^{(n)}(d,k)'$ of $S_{m,r}^{(n)}(d,k)$ for each $m \ge 2$, $1 \le d \le H_n$, $0 \le k < g_n$, and $1 \le r \le K_n$, all defined on a common probability space with $\{Y_i\}_{i\in\mathbb{N}}$ and $\{Y_i'\}_{i\in\mathbb{N}}$ from the previous subsection. We assume that the random vectors

$$\big(S_{m,r}^{(n)}(d_m,k_m), S_{m,r}^{(n)}(d_m,k_m)'\big), \qquad m \in \mathbb{N},\ 1 \le r \le K_n,$$

with $1 \le d_m \le H_n$ and $0 \le k_m < g_n$ for each $m \in \mathbb{N}$, are mutually independent, and that they are also independent of

$$\big\{ (Y_j, Y_j') : j \notin \{ i_{m,r}, i_{m,r} + (2d_m)2^{k_m} \} \text{ for any } m \ge 2 \text{ and } 1 \le r \le K_n \big\}.$$

If (i,j) is a (ψ_0,ψ_1)-good pair in block $B_m^{(n)}$, $m \ge 2$, we have, similarly to (6.24),

$$\min\big\{ \mathbb{P}[Y_i + Y_j = i], \mathbb{P}[Y_i + Y_j = j] \big\} \ge \frac{\psi_0\psi_1}{mL_n}.$$

This allows define joint distributions of $S_{m,r}^{(n)}(d,k)$ and $S_{m,r}^{(n)}(d,k)'$ by

$$\begin{aligned}
\mathbb{P}\Big[\big(S_{m,r}^{(n)}(d,k), S_{m,r}^{(n)}(d,k)'\big) = (i_{m,r}, i_{m,r} + 2^k)\Big] &:= \frac{\psi_0\psi_1}{mL_n} \wedge \frac{1}{2} > 0, \\
\mathbb{P}\Big[\big(S_{m,r}^{(n)}(d,k), S_{m,r}^{(n)}(d,k)'\big) = (i_{m,r} + 2^k, i_{m,r})\Big] &:= \frac{\psi_0\psi_1}{mL_n} \wedge \frac{1}{2} > 0, \\
\mathbb{P}\Big[S_{m,r}^{(n)}(d,k) = S_{m,r}^{(n)}(d,k)'\Big] &:= 1 - 2\Big(\frac{\psi_0\psi_1}{mL_n} \wedge \frac{1}{2}\Big).
\end{aligned} \tag{6.53}$$

Let

$$\tau_m^{(n)} := \tau_m^{(n)}(d,k) := \inf\big\{ 1 \le r \le K_n : S_{m,r}^{(n)}(d,k) \ne S_{m,r}^{(n)}(d,k)' \big\},$$

where we set $\tau_m^{(n)} := \infty$ if $S_{m,r}^{(n)}(d,k) = S_{m,r}^{(n)}(d,k)'$ for each $1 \le r \le K_n$. Then, recalling the definitions of $A_-^{(n)}$, $A_+^{(n)}$ and $A^{(n)}$ at the end of Subsection 6.4.1, we define for each pair $(j,k) \in A^{(n)} \times \{0,\dots,g_n\}$ the random variables

$$\begin{aligned}
S_m^{(n)}(j,k) :=\ & \mathbf{1}\big\{j \in A_+^{(n)} \setminus \{1\} \text{ and } k < g_n\big\}\Big(\sum_{r=1}^{K_n} S_{m,r}\big(\tfrac{j-1}{2},k\big) + \sum_{i\in R_m^{(n)}} Y_i\Big) \\
& + \mathbf{1}\big\{j \in A_-^{(n)} \text{ and } k < g_n\big\}\Big(\sum_{r=1}^{K_n} S_{m,r}\big(\tfrac{1-j}{2},k\big) + \sum_{i\in R_m^{(n)}} Y_i\Big) \\
& + \mathbf{1}\big\{j \in \{0,1\} \text{ or } k = g_n\big\} \sum_{i\in B_m^{(n)}} Y_i\,, \\
S_m^{(n)}(j,k)' :=\ & \mathbf{1}\big\{j \in A_+^{(n)} \setminus \{1\} \text{ and } k < g_n\big\} \\
& \times \Big(\sum_{r=1}^{\tau_m^{(n)}\wedge K_n} S_{m,r}\big(\tfrac{j-1}{2},k\big)' + \sum_{r=\tau_m^{(n)}+1}^{K_n} S_{m,r}\big(\tfrac{j-1}{2},k\big) + \sum_{i\in R_m^{(n)}} Y_i\Big) \\
& + \mathbf{1}\big\{k \in A_-^{(n)} \text{ and } k < g_n\big\} \\
& \times \Big(\sum_{r=1}^{\tau_m^{(n)}\wedge K_n} S_{m,r}\big(\tfrac{1-j}{2},k\big)' + \sum_{r=\tau_m^{(n)}+1}^{K_n} S_{m,r}\big(\tfrac{1-j}{2},k\big) + \sum_{i\in R_m^{(n)}} Y_i\Big) \\
& + \mathbf{1}\big\{j \in \{0,1\} \text{ or } k = g_n\big\} \sum_{i\in B_m^{(n)}} Y_i\,,
\end{aligned}$$

and also introduce

$$\begin{aligned}
U_m^{(n)}(j,k) &:= S_m^{(n)}(j,k) - S_m^{(n)}(j,k)'\,, \\
I_m^{(n)}(j,k) &:= \mathbf{1}\{U_m^{(n)}(j,k) \neq 0\}\,.
\end{aligned}$$

Lemma 6.22. *(i) The distributions of $S_m^{(n)}(j,k)$ and $S_m^{(n)}(j,k)'$ do not depend on the choice of (j,k). More precisely,*

$$\mathscr{L}\big(S_m^{(n)}(j,k)\big) = \mathscr{L}\big(S_m^{(n)}(j,k)'\big) = \mathscr{L}\big(T_{B_m^{(n)}}\big) \quad \textit{for all } (j,k) \in A^{(n)} \times \{0,\dots,g_n\},$$

with $T_{B_m^{(n)}}$ defined in (6.12).

(ii) Let $j \in A^{(n)} \setminus \{0,1\}$ and $0 \le k < g_n$. The probability of a U-jump in block $B_m^{(n)}$, that is an event of the form $U_m^{(n)}(j,k) \neq 0$, is strictly positive and does not depend on (j,k). Indeed, it is equal to

$$p_m^{(n)} := \mathbb{P}\big[I_m^{(n)}(j,k) = 1\big] = 1 - \bigg(1 - 2\Big(\frac{\psi_0\psi_1}{mL_n} \wedge \frac{1}{2}\Big)\bigg)^{K_n}.$$

Moreover,

$$\tilde{p}_m^{(n)} - (\tilde{p}_m^{(n)})^2 \le p_m^{(n)} \le \tilde{p}_m^{(n)},$$

where

$$\frac{\psi_0\psi_1}{4m} \wedge K_n \le \tilde{p}_m^{(n)} := 2K_n\Big(\frac{\psi_0\psi_1}{mL_n} \wedge \frac{1}{2}\Big) \le \frac{\psi_0\psi_1}{4m}.$$

If $j \in \{0,1\}$ or if $k = g_n$, we have $I_m^{(n)}(j,k) = 1$.

(iii) If $j \in A_+^{(n)}$, we have $U_m^{(n)}(j,k) \in \{-(j-1)2^k, 0, (j-1)2^k\}$, and if $k < g_n$, it follows that

$$\mathbb{P}\big[U_m^{(n)}(j,k) = -(j-1)2^k \;\big|\; I_m^{(n)}(j,k) = 1\big] = \frac{1}{2}$$
$$\mathbb{P}\big[U_m^{(n)}(j,k) = (j-1)2^k \;\big|\; I_m^{(n)}(j,k) = 1\big] = \frac{1}{2}.$$

If $k = g_n$, it follows that $U_m^{(n)}(j,k) = 0$.

If $j \in A_-^{(n)}$, we have $U_m^{(n)}(j,k) \in \{-(j+1)2^k, 0, (j+1)2^k\}$, and if $k < g_n$, it follows that

$$\mathbb{P}\big[U_m^{(n)}(j,k) = -(j+1)2^k \;\big|\; I_m^{(n)}(j,k) = 1\big] = \frac{1}{2}$$
$$\mathbb{P}\big[U_m^{(n)}(j,k) = (j+1)2^k \;\big|\; I_m^{(n)}(j,k) = 1\big] = \frac{1}{2}.$$

If $k = g_n$, we have $U_m^{(n)}(j,k) = 0$.

Finally, if $j = 0$, it follows that $U_m^{(n)}(j,k) = 0$ again.

Proof. The lemma follows with arguments as used in the proof of Lemma 6.12. □

6.4.4 Jump counts

We keep up the assumptions from the beginning of the previous subsection, and recall the definition of $U_1^{(n)}$ and $\tau_1^{(n)}$ in (6.52) and (6.50), respectively. We define

$$J_1^{(n)} := 0 \qquad \text{and} \qquad J_m^{(n)} := J_{m-1}^{(n)} + I_m^{(n)}(U_1^{(n)}, J_{m-1}^{(n)}) \qquad \text{for each } m \ge 2.$$

The random variable $J_m^{(n)}$ counts the number of U-jumps in consecutive blocks starting with the second block $B_2^{(n)}$ up to $B_m^{(n)}$. The recursive definition of the jump counters and Lemma 6.22 (ii) yield

$$0 = J_1^{(n)} \le J_2^{(n)} \le \dots \le J_{m-1}^{(n)} \le J_m^{(n)} \le (m-1) \wedge g_n.$$

Let

$$\mathcal{E}^{(n)} := \{\tau_1^{(n)} = 1, U_1^{(n)} = -1\} \mathbin{\mathring{\cup}} \{1 < \tau_1^{(n)} < \infty\}. \tag{6.54}$$

Lemma 6.23. *Assume that*

$$\frac{\psi_0\psi_1}{2L_n} \le \frac{1}{2}\,.$$

Then we have for every $x > 0$ every $m \ge 2$

$$\mathbb{P}\big[J_m^{(n)} < g_n,\, \mathcal{E}^{(n)}\big] \le \mathrm{e}^{xg_n} \exp\big((1-\mathrm{e}^{-x})\psi_0^2\psi_1^2\pi^2/96\big)\left(\frac{2}{m+1}\right)^{\psi_0\psi_1(1-\mathrm{e}^{-x})/4}.$$

Proof. The proof is very similar to the proof of Lemma 6.14. It follows from Lemma 6.21 (iii) that on the event $\mathcal{E}^{(n)}$ we have $U_1^{(n)} \in A^{(n)} \setminus \{0,1\}$. Then from Lemma 6.21 (ii) we conclude that

$$\mathbb{P}\big[J_m^{(n)} < g_n,\, \mathcal{E}^{(n)}\big] = \mathbb{P}\Big[\sum\nolimits_{r=2}^{m} I_r^{(n)} < g_n,\, \mathcal{E}^{(n)}\Big] \le \mathbb{P}\Big[\sum\nolimits_{r=2}^{m} I_r^{(n)} < g_n\Big]\,,$$

where $I_2^{(n)}, \dots, I_m^{(n)}$ are independent Bernoulli distributed random variables with expectations

$$\mathbb{E}I_r^{(n)} = 1 - \left(1 - \frac{2\psi_0\psi_1}{mL_n}\right)^{K_n} \qquad \text{for } 2 \le r \le m.$$

The lemma follows by invoking Chernov bounds and Lemma 6.21 (ii) once more. □

6.4.5 Total variation bounds

In this subsection, we require the assumptions of both Subsection 6.4.2 and Subsection 6.4.3 to hold; that is $\tilde{a}_n < 2H_n - 1 \le L_n$ and $K_n = 2^{l_n-3} \ge 1$, $H_n = 2^{h_n-1} \ge 1$, $g_n = l_n - h_n \ge 1$. Moreover, assume that $(\psi_0 \wedge \psi_1)/N \le 1/2$ and that $\psi_0\psi_1/(2L_n) \le 1/2$, so that we can apply Lemma 6.21 (i) and Lemma 6.23.

For each $m \ge 2$ we define

$$T'_{B_m^{(n)}} := S_m^{(n)}(U_1^{(n)}, J_{m-1}^{(n)}) \quad \text{and} \quad T''_{B_m^{(n)}} := S_m^{(n)}(U_1^{(n)}, J_{m-1}^{(n)})'\,.$$

That is, we replace in $S_m^{(n)}(j,k)$ and $S_m^{(n)}(j,k)'$ the fixed number $j \in A^{(n)}$ with the random variable $U_1^{(n)}$, which, by Lemma 6.22 (iii) is concentrated on $A^{(n)}$, and we replace $0 \le k \le g_n$ by the random number of jumps $J_{m-1}^{(n)}$ that have occurred up to the preceding block number $m-1$. Recall that $T'_{B_1^{(n)}}$ and $T''_{B_1^{(n)}}$ have been introduced in (6.51) already.

These random variables are copies of (6.12), as follows from the lemma below.

Lemma 6.24. *The sequences*

$$\{T'_{B_m^{(n)}}\}_{m\in\mathbb{N}} \qquad \textit{and} \qquad \{T''_{B_m^{(n)}}\}_{m\in\mathbb{N}}$$

consist of independent random variables. Moreover,

$$\mathscr{L}\big(T'_{B_m^{(n)}}\big) = \mathscr{L}\big(T''_{B_m^{(n)}}\big) = \mathscr{L}\big(T_{B_m^{(n)}}\big) \qquad \textit{for all } m \in \mathbb{N}.$$

Proof. The proof is similar to the one of Lemma 6.15; here we use the independence of $(S_m^{(n)}(j,k), S_m^{(n)}(j,k))'$, $m \geq 2$, and $(U_1^{(n)}, J_{m-1}^{(n)})$ for each pair $(j,k) \in A^{(n)} \times \{0,\dots,g_n\}$. □

Recall from (6.41) that, for some fixed $0 < \varepsilon < 1$, $L_n := L_n(\varepsilon)$ is the length of a block $B_m^{(n)}$. Then

$$M_n := M_n(\varepsilon) := \left\lfloor \frac{n}{L_n(\varepsilon)} \right\rfloor$$

is the largest index of a block contained entirely in $\{1,\dots,n\}$, or zero.

We invoke Lemma 6.5 and conclude that

$$\begin{aligned} d_{\mathrm{TV}}\Big(\mathscr{L}\big(T_{\tilde{a}_n,n}\big), \mathscr{L}\big(T_{\tilde{a}_n,n}+1\big)\Big) &\leq \mathbb{P}[\tau^{(n)} > M_n] \\ &\leq \mathbb{P}[\tau^{(n)} > M_n,\, \tau_1^{(n)} = \infty] && (6.55\text{a}) \\ &\quad + \mathbb{P}\big[\tau^{(n)} > M_n,\, \tau_1^{(n)} = 1, U_1^{(n)} = 1\big] && (6.55\text{b}) \\ &\quad + \mathbb{P}\big[\tau^{(n)} > M_n,\, J_{M_n}^{(n)} < g_n,\, \mathcal{E}^{(n)}\big] && (6.55\text{c}) \\ &\quad + \mathbb{P}\big[\tau^{(n)} > M_n,\, J_{M_n}^{(n)} = g_n,\, \mathcal{E}^{(n)}\big], && (6.55\text{d}) \end{aligned}$$

recalling the definition of $\tau_1^{(n)}$ in (6.50) and of $\mathcal{E}^{(n)}$ in (6.54). Note that in Lemma 6.5 we use blocks $B_m^{(n)}$ slightly different from those used in this section (cf. the definitions (6.10) and (6.38), (6.39)). This, however, does not change the conclusion of the lemma at all.

The first of the four summands, (6.55a), is bounded using Lemma 6.21 (i); the second summand, (6.55b), is zero if $M_n \geq 2$. The bounds for the last two summands, (6.55c) and (6.55d), are given in the next two lemmas.

Lemma 6.25. *Let $0 < \varepsilon < 1$ and $M_n(\varepsilon) \geq 2$. It follows that*

$$\mathbb{P}[J_{M_n}^{(n)} < g_n,\, \mathcal{E}^{(n)}] \leq \rho(\psi_0,\psi_1)\Big(\frac{\bar{a}_n}{n}\Big)^{\psi_0\psi_1/8 - \varepsilon(1/2+\psi_0\psi_1/8)},$$

where $\rho(\psi_0,\psi_1) := \exp(\psi_0^2\psi_1^2\pi^2/192)2^{1+\psi_0\psi_1/8}$.

Proof. We invoke Lemma 6.23 with $m := M_n$ and $x := \log 2$, which then yields

$$\mathbb{P}\big[J_{M_n}^{(n)} < g_n,\, \mathcal{E}^{(n)}\big] \leq 2^{g_n} \exp\big(\psi_0^2\psi_1^2\pi^2/192\big)\left(\frac{2}{M_n+1}\right)^{\psi_0\psi_1/8}.$$

For a fixed $0 < \varepsilon < 1$ we recall from (6.41) and (6.42) that

$$M_n + 1 = M_n(\varepsilon) + 1 \geq \frac{n}{L_n(\varepsilon)} \geq \Big(\frac{n}{\bar{a}_n}\Big)^{1-\varepsilon},$$

and from (6.47), (6.42) and (6.46) that

$$2^{g_n} = 2^{g_n(\varepsilon)} = \frac{L_n(\varepsilon)}{2H_n(\varepsilon)} \leq 2\Big(\frac{n}{\bar{a}_n}\Big)^{\varepsilon/2}.$$

This proves the lemma. □

Lemma 6.26. *Let $0 < \varepsilon < 1$ and $M_n(\varepsilon) \geq 2$. Then we have*

$$\mathbb{P}[\tau^{(n)} > M_n,\, J_{M_n}^{(n)} < g_n,\, \mathcal{E}^{(n)}] \leq \frac{1}{2^{g_n}} \leq 2\Big(\frac{\bar{a}_n}{n}\Big)^{\varepsilon/2}.$$

Proof. Recalling the definition of $\mathcal{E}^{(n)}$ in (6.54), we write

$$\begin{aligned}\mathbb{P}[\tau^{(n)} > M_n,\, J_{M_n}^{(n)} < g_n,\, \mathcal{E}^{(n)}] = &\sum_{j \in A_+^{(n)} \setminus \{1\}} \mathbb{P}[\tau^{(n)} > M_n,\, J_{M_n}^{(n)} < g_n,\, U_1^{(n)} = j] \\ &+ \sum_{j \in A_-^{(n)}} \mathbb{P}[\tau^{(n)} > M_n,\, J_{M_n}^{(n)} < g_n,\, U_1^{(n)} = j]\end{aligned}$$

With arguments similar to those used in the proof of Lemma 6.17 we can show that

$$\mathbb{P}[\tau^{(n)} > M_n,\, J_{M_n}^{(n)} < g_n,\, U_1^{(n)} = j] \leq \frac{1}{2^{g_n}} \mathbb{P}[U_1^{(n)} = j]$$

for every $j \in A_+^{(n)} \cup A_-^{(n)} \setminus \{1\}$. This yields the first inequality of Lemma 6.26. The second follows from the definition of g_n in (6.47), (6.42) and (6.46). □

We collect the results of this section in the following theorem.

Theorem 6.27. *Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}$ -valued random variables that satisfies Assumption 6.19 for $N \in \mathbb{N}$ and $\psi_0, \psi_1 > 0$. Assume that $(\psi_0 \wedge \psi_1)/N \leq 1/2$. Let*

$$\varepsilon := \frac{\psi_0\psi_1}{\psi_0\psi_1 + 4(\psi_0 \wedge \psi_1 \wedge 1) + 4}.$$

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of integers with $0 \le a_n < n$ for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ such that $\tilde{a}_n < 2H_n(\varepsilon) - 1 \le L_n(\varepsilon)$, $K_n(\varepsilon) \ge 1$, $H_n(\varepsilon) \ge 1$, $g_n(\varepsilon) \ge 1$, $M_n(\varepsilon) \ge 1$ and

$$\frac{\psi_0\psi_1}{2L_n(\varepsilon)} \le \frac{1}{2}$$

it follows that

$$\begin{aligned} d_{\mathrm{TV}}\big(&\mathscr{L}(T_{a_n,n}), \mathscr{L}(T_{a_n,n}+1)\big) \\ &\le (\rho(\psi_0,\psi_1) + 12^{\psi_0\wedge\psi_1} + 2)\Big(\frac{\bar{a}_n}{n}\Big)^{(\psi_0\wedge\psi_1\wedge 1)\psi_0\psi_1/(2\psi_0\psi_1+8(\psi_0\wedge\psi_1\wedge 1)+8)}, \end{aligned}$$

with $\rho(\psi_0,\psi_1)$ as in Lemma 6.25. In particular, if $a_n = \mathrm{o}(n)$, the total variation distance between $\mathscr{L}(T_{a_n,n})$ and $\mathscr{L}(T_{a_n,n}+1)$ converges to 0.

Proof. From (6.55) we conclude with Lemma 6.21 (i), 6.25 and 6.26 that

$$\begin{aligned} d_{\mathrm{TV}}\big(\mathscr{L}(T_{\tilde{a}_n,n}), \mathscr{L}(T_{\tilde{a}_n,n}+1)\big) \le (12^{\psi_0\wedge\psi_1} + 2)\Big(\frac{\bar{a}_n}{n}\Big)^{(\psi_0\wedge\psi_1\wedge 1)\varepsilon/2} \\ + \rho(\psi_0,\psi_1)\Big(\frac{\bar{a}_n}{n}\Big)^{\psi_0\psi_1/8-\varepsilon(1/2+\psi_0\psi_1/8)}. \end{aligned}$$

Our choice of ε maximises the exponents. Because $a_n \le \tilde{a}_n$, we also have

$$d_{\mathrm{TV}}\big(\mathscr{L}(T_{a_n,n}), \mathscr{L}(T_{a_n,n}+1)\big) \le d_{\mathrm{TV}}\big(\mathscr{L}(T_{\tilde{a}_n,n}), \mathscr{L}(T_{\tilde{a}_n,n}+1)\big);$$

the theorem follows. □

A Appendix

A.1 Basic definitions, notations and properties

A.1.1 Standard distributions on $\mathbb{Z}_+$

The *Poisson distribution* with expectation $\lambda \geq 0$ is denoted by $\mathrm{Po}(\lambda)$. Recall that point probabilities of a $\mathrm{Po}(\lambda)$-distributed random variable Z are defined as

$$\mathbb{P}[Z = k] := \mathrm{e}^{-\lambda}\frac{\lambda^k}{k!} \qquad \text{for all } k \in \mathbb{Z}_+.$$

We use the convention that $\mathrm{Po}(0)$ is equal to the Dirac measure δ_0 at 0.

The *binomial distribution* of $n \in \mathbb{N}$ independent trials with success probability $0 \leq p \leq 1$ is denoted by $\mathrm{Bin}(n,p)$. Point probabilities of a $\mathrm{Bin}(n,p)$-distributed random variable Z are

$$\mathbb{P}[Z = k] := \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{for all } 0 \leq k \leq n.$$

In particular, we use $\mathrm{Be}(p) := \mathrm{Bin}(1,p)$ to denote the *Bernoulli distribution*, and we use the convention that $\mathrm{NB}(0,p) = \delta_0$.

The *negative binomial distribution*, which counts the number of failures, each with probaility $0 \leq p < 1$ up to success number $n \in \mathbb{N}$, is denoted by $\mathrm{NB}(n,p)$. Point probabilities of a $\mathrm{NB}(n,p)$-distributed random variable Z are defined as

$$\mathbb{P}\big[Z = k\big] := \binom{n+k-1}{k} p^k (1-p)^n \qquad \text{for } k \in \mathbb{Z}_+.$$

We use $\mathrm{Ge}(p) := \mathrm{NB}(1,p)$ to denote the *geometric distribution*, and we set $\mathrm{NB}(0,p) := \delta_0$.

A.1.2 Probability metrics

We refer to Rachev (1991) and Barbour et al. (1992, Appendix A.1) for an overview on probability metrics. Here, we only give a short account. Let X and Y be random variables taking values in the measurable space $(\mathcal{X}, \mathcal{A}_{\mathcal{X}})$.

The *total variation distance* d_{TV} between the distributions X and Y can be defined in several equivalent forms

$$\begin{aligned} d_{\mathrm{TV}}\big(\mathscr{L}(X),\mathscr{L}(Y)\big) &= \sup\nolimits_{f\in\mathcal{F}_{\mathrm{TV}}}\big|\mathbb{E}f(X)-\mathbb{E}f(Y)\big| \\ &= \sup\nolimits_{f\in\mathcal{F}^*_{\mathrm{TV}}}\big|\mathbb{E}f(X)-\mathbb{E}f(Y)\big| \\ &= \sup\nolimits_{A\in\mathcal{A}_{\mathcal{X}}}\big|\mathbb{P}[X\in A]-\mathbb{P}[Y\in A]\big|\,, \end{aligned}$$

where

$$\begin{aligned} &\mathcal{F}_{\mathrm{TV}} := \big\{f:\mathcal{X}\to\mathbb{R} : \|f\|\le 1/2\big\}\,, \\ &\mathcal{F}^*_{\mathrm{TV}} := \big\{f:\mathcal{X}\to\mathbb{R} : \exists\, A\in\mathcal{A}_{\mathcal{X}} \text{ such that } f(\cdot)=\mathbf{1}\{\cdot\in A\}\big\}\,. \end{aligned}$$

If $\mathcal{X}$ is discrete, we also have

$$d_{\mathrm{TV}}\big(\mathscr{L}(X),\mathscr{L}(Y)\big) = \frac{1}{2}\sum_{i=0}^{\infty}\big|\mathbb{P}[X=i]-\mathbb{P}[Y=i]\big|\,. \tag{A.1}$$

The total variation distance can also be represented in the form

$$d_{\mathrm{TV}}\big(\mathscr{L}(X),\mathscr{L}(Y)\big) = \min\mathbb{P}\big[\tilde{X}\ne\tilde{Y}\big]\,,$$

where the minimum is taken over all couplings $(\tilde{X},\tilde{Y})$ of X and Y.

Now, let $\mathcal{X}\in\{\mathbb{R},\mathbb{R}_+\}$ and let $\mathcal{A}_{\mathcal{X}}$ be the Borel σ-algebra. The *Wasserstein distance* d_{W} between the distributions of X and Y is defined by

$$d_{\mathrm{W}}\big(\mathscr{L}(X),\mathscr{L}(Y)\big) := \sup\nolimits_{f\in\mathcal{F}_{\mathrm{W}}}\big|\mathbb{E}f(X)-\mathbb{E}f(Y)\big|\,,$$

where

$$\mathcal{F}_{\mathrm{W}} := \big\{f:\mathcal{X}\to\mathbb{R} : |f(u)-f(v)|\le|u-v| \text{ for all } u,v\in\mathcal{X}\big\}\,.$$

The *bounded Wasserstein distance* d_{BW} between the distributions of X and Y is defined by

$$d_{\mathrm{W}}\big(\mathscr{L}(X),\mathscr{L}(Y)\big) := \sup\nolimits_{f\in\mathcal{F}_{\mathrm{BW}}}\big|\mathbb{E}f(X)-\mathbb{E}f(Y)\big|\,,$$

where

$$\mathcal{F}_{\mathrm{BW}} := \big\{f:\mathcal{X}\to\mathbb{R} : f\in\mathcal{F}_{\mathrm{W}} \text{ and } \|f\|\le 1/2\big\}\,.$$

A.1.3 The harmonic series and the Riemann Zeta function

To bound the harmonic series we apply

$$\log(n+1) \le \sum_{i=1}^{n} \frac{1}{i} \le 1 + \log n \qquad \text{for all } n \in \mathbb{N}, \tag{A.2}$$

and

$$\log\Big(\frac{n+1}{k+1}\Big) \le \sum_{i=k+1}^{n} \frac{1}{i} \le \log\Big(\frac{n}{k}\Big) \qquad \text{for all } k, n \in \mathbb{N},\ k < n. \tag{A.3}$$

The *Euler constant* is

$$\gamma := \lim_{n\to\infty}\Bigg(\sum_{i=1}^{n} \frac{1}{i} - \log n\Bigg) \approx 0.57722\,.$$

We use γ exclusively to denote this constant.

The *Riemann Zeta function* is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for all } s \in \mathbb{R},\ s > 0,$$

and ζ is used exclusively to denote this function. Note that

$$\zeta(2) = \frac{\pi^2}{6} \tag{A.4}$$

A.1.4 Slowly varying sequences

The standard reference for slow variation is Bingham et al. (1989).

Definition A.1. A positive sequence $\{c_n\}_{n\in\mathbb{N}}$ is *slowly varying (at infinity)* if

$$\lim_{n\to\infty} \frac{c_{\lfloor \lambda n \rfloor}}{c_n} = 1 \qquad \text{for all } \lambda > 0.$$

Theorem A.2 (Bingham et al. (1989, Theorem 1.9.7))**.** *A positive sequence $\{c_n\}_{n\in\mathbb{N}}$ is slowly varying if and only if it may be written in the form*

$$c_n = C_n \exp\Bigg(\sum_{i=1}^{n} \frac{\delta_i}{i}\Bigg) \qquad \textit{for all } n \in \mathbb{N},$$

where $\{C_n\}_{n\in\mathbb{N}}$ and $\{\delta_i\}_{i\in\mathbb{N}}$ are sequences that satisfy

$$\lim_{n\to\infty} C_n = C \quad \textit{for some } C \in (0,\infty), \qquad \textit{and} \qquad \lim_{i\to\infty} \delta_i = 0\,.$$

Theorem A.3 (Bingham et al. (1989, Theorem 1.4.1))**.** *Let $\{c_n\}_{n\in\mathbb{N}}$ be a positive sequence that satisfies*

$$\lim_{n\to\infty} \frac{c_{\lfloor \lambda n\rfloor}}{c_n} = 1$$

for all λ in a subset in $(0,\infty)$ of positive Lebesgue measure. Then $\{c_n\}_{n\in\mathbb{N}}$ is slowly varying.

Proposition A.4 (Bingham et al. (1989, Proposition 1.3.6 and Lemma 1.9.6))**.** *Let $\{c_n\}_{n\in\mathbb{N}}$ be a slowly varying sequence. Then we have*

(i)

$$\lim_{n\to\infty} \frac{\log c_n}{\log n} = 0\,, \tag{A.5}$$

(ii)

$$\lim_{n\to\infty} n^{\epsilon} c_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{c_n}{n^{\epsilon}} = 0 \qquad \textit{for all } \varepsilon > 0, \tag{A.6}$$

(iii)

$$\lim_{n\to\infty} \frac{c_{n-1}}{c_n} = 1\,. \tag{A.7}$$

In view of Theorem A.2, (A.7) can be generalized immediately to

$$\lim_{n\to\infty} \frac{c_{k_n}}{c_n} = 1 \tag{A.8}$$

for any integer sequence $\{k_n\}_{n\in\mathbb{N}}$ with $k_n \sim n$. Indeed, we have, recalling (A.3),

$$\left(\frac{k_n}{n}\right)^{\|\delta\|} \le \exp\left(\sum_{i=k_n+1}^{n} \frac{\delta_i}{i}\right) \le \left(\frac{n}{k_n}\right)^{\|\delta\|},$$

where $\|\delta\| := \sup_{i\in\mathbb{N}}|\delta_i|$.

A.1.5 Uniformly distributed modulo 1 sequences

For details on the theory briefly outlined in this section we refer to Kuipers and Niederreiter (1974).

Definition A.5 (Kuipers and Niederreiter (1974, Chapter 1, Definition 1.1))**.** A sequence $\{x_j\}_{j\in\mathbb{N}}$ of real numbers is *uniformly distributed modulo* 1 *(u. d. mod* 1*)* if

$$\lim_{n\to\infty} \frac{1}{n}\Big|\big\{1 \le j \le n : \alpha \le x_j - \lfloor x_j \rfloor < \beta\big\}\Big| = \beta - \alpha \qquad \text{for all } 0 \le \alpha < \beta \le 1.$$

Theorem A.6 (Weyl criterion, Kuipers and Niederreiter (1974, Chapter 1, Theorem 2.1))**.** *A sequence* $\{x_j\}_{j\in\mathbb{N}}$ *of real numbers is u. d. mod* 1 *if and only if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathrm{e}^{2\pi i l x_j}=0 \qquad \textit{for all } l\in\mathbb{Z}\setminus\{0\}.$$

Example A.7 (Kuipers and Niederreiter (1974, Chapter 1, Example 2.1))**.** Let y be an irrational number. Then the sequence $\{jy\}_{j\in\mathbb{N}}$ is u. d. mod 1.

Definition A.8 (Kuipers and Niederreiter (1974, Chapter 2, Definition 1.2))**.** For a finite sequence $x_1,\dots,x_n$ of real numbers the *discrepancy* is defined as

$$D_n^* := \sup_{0<\alpha\le 1}\left|\frac{1}{n}\left|\left\{1\le j\le n : x_j-\lfloor x_j\rfloor\le\alpha\right\}\right|-\alpha\right|.$$

For an infinite sequence $\mathfrak{x} := \{x_j\}_{j\in\mathbb{N}}$, D_n^* is defined using the first n elements of $\mathfrak{x}$.

The discrepancy measures the deviation of a sequence from an "ideal" uniform distribution modulo 1. Indeed, it can be shown that a sequence is u. d. mod 1 if and only if its discrepancy D_n^* converges to 0 as $n\to\infty$ (Kuipers and Niederreiter, 1974, Chapter 2, Corollary 1.1).

Theorem A.9 (Theorem of Erdős-Turán, Kuipers and Niederreiter (1974, Chapter 2, Theorem 2.5))**.** *Let* $x_1,\dots,x_n$ *be a finite sequence of real numbers with discrepancy* D_n^**. Then it follows that*

$$D_n^*\le\frac{6}{r+1}+\frac{4}{\pi}\sum_{l=1}^{r}\left(\frac{1}{l}-\frac{1}{r+1}\right)\left|\frac{1}{n}\sum_{j=1}^{n}\mathrm{e}^{2\pi i l x_j}\right| \qquad \textit{for all } r\in\mathbb{N}.$$

Theorem A.10 (Koksma inequality, Kuipers and Niederreiter (1974, Chapter 2, Theorem 5.1))**.** *Let* $f:[0,1]\to\mathbb{R}$ *be a function of bounded variation* $V(f)$*. Suppose we are given* n *points* $x_1,\dots,x_n\in[0,1)$ *with discrepancy* D_n^**. Then we have*

$$\left|\frac{1}{n}\sum_{j=1}^{n}f(x_j)-\int_0^1 f(t)\,dt\right|\le V(f)D_n^*\,.$$

We denote the distance of a real number t to its nearest integer as

$$\langle t\rangle := \min_{r\in\mathbb{Z}}|t-r|\,.$$

Definition A.11 (Kuipers and Niederreiter (1974, Chapter 2, Definition 3.4))**.** Let y be an irrational number, and let

$$\mathcal{R}(y) := \left\{ \tau \in \mathbb{R} : \liminf_{n\to\infty} n^{\tau} \langle ny \rangle = 0 \right\}.$$

The number y is of *irrational type* η if $\eta = \sup \mathcal{R}(y)$.

It follows from Dirichlet's theorem that the type η of an irrational number y satisfies $1 \le \eta \le \infty$. If $\eta < \infty$, we say that y is of *finite type*. Algebraic irrationals are of type $\eta = 1$. Irrationals number of type $\eta = \infty$ are called *Liouville numbers*.

The following lemma can be used, together with Theorem A.9 to obtain explicit bounds on the discrepancy of the sequence $\{yj\}_{j\in\mathbb{N}}$, for an irrational number y of finite type.

Lemma A.12 (Kuipers and Niederreiter (1974, Chapter 2, Lemma 3.3 and Example 3.2))**.** *Let y be an irrational number of finite type η. For every $\varepsilon > 0$ there is a constant $c(\eta, \varepsilon) > 0$ such that*

$$\sum_{j=1}^{n} \frac{1}{j\langle jy \rangle} \le c(\eta,\varepsilon)\, n^{\eta-1+\varepsilon} \qquad \textit{for all } n \in \mathbb{N}.$$

In particular, if $\eta = 1$, there is a constant $c > 0$ such that

$$\sum_{j=1}^{n} \frac{1}{j\langle jy \rangle} \le c\,(\log n)^2 \qquad \textit{for all } n \in \mathbb{N}.$$

A.2 Technical appendix for Section 2

A.2.1 Bounds used in combination with condition UC

Let the assumptions and definitions from the beginning of Subsection 2.1.1 be in force. The following technical lemma is used in combination with condition UC in various situations.

Lemma A.13. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of $\mathbb{Z}_+$-valued random variables such that $\mathbb{E}Z_i < \infty$ for all $i \in \mathbb{N}$, let and $\mathfrak{r} := \{r_i\}_{i\in\mathbb{N}}$ a sequence of natural numbers* *such that* (2.2) *holds. Then we have for every* $i \in \mathbb{N}$

$$\mathbb{E}Z_i\big|\varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big| \le \mathbb{E}Z_i - \mathbb{P}[Z_i = 1] \le \mathbb{E}Z_i\Big(\big|\varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big| + \mathbb{E}Z_i\Big)\,, \tag{A.9}$$

$$\mathbb{P}[Z_i \ge 2] \le \frac{\mathbb{E}Z_i}{2}\Big(\big|\varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big| + \mathbb{E}Z_i\Big)\,, \tag{A.10}$$

$$\big|1 - \mathbb{P}[Z_i = 0] - \mathbb{E}Z_i\big| \le \frac{3\mathbb{E}Z_i}{2}\Big(\big|\varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big| + \mathbb{E}Z_i\Big)\,, \tag{A.11}$$

$$\mathbb{P}[Z_i = k] \le \frac{\mathbb{E}Z_i}{k}\Big(\big|\varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big| + \mathbb{E}Z_i\Big) \qquad \text{for all } k \ge 2. \tag{A.12}$$

Moreover,

$$d_{\mathrm{TV}}\big(\mathscr{L}(Z_i), \mathrm{Be}(1 \wedge \mathbb{E}Z_i)\big) \le \frac{3\mathbb{E}Z_i}{2}\Big(\big|\varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big| + \mathbb{E}Z_i\Big) + (\mathbb{E}Z_i - 1)^+\,. \tag{A.13}$$

Proof. For all $i \in \mathbb{N}$ we have

$$\begin{aligned} -1 \le \varepsilon_{i1}(i\mathbb{E}Z_i, r_i) \le E_i(i\mathbb{E}Z_i, r_i) &:= \sum_{k=1}^{\infty} \varepsilon_{ik}(i\mathbb{E}Z_i, r_i) \\ &\le \sum_{k=1}^{\infty} k\varepsilon_{ik}(i\mathbb{E}Z_i, r_i) = 0\,. \end{aligned} \tag{A.14}$$

First, let $\mathbb{E}Z_i > 0$. It follows, because $-1 \le \varepsilon_{i1}(i\mathbb{E}Z_i, r_i) \le 0$ from (A.14), that

$$\begin{aligned} \mathbb{P}[Z_i = 1] &= \mathbb{P}[Z_{i1} = 1]\mathbb{P}[Z_{i1} = 0]^{r_i - 1} \\ &= \mathbb{E}Z_i\big(1 + \varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big)\Big(1 - \frac{\mathbb{E}Z_i}{r_i}\big(1 + E_i(i\mathbb{E}Z_i, r_i)\big)\Big)^{r_i - 1} \\ &\le \mathbb{E}Z_i\big(1 + \varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big) \\ &= \mathbb{E}Z_i - \mathbb{E}Z_i\big|\varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big|\,, \end{aligned}$$

and, by the Bernoulli inequality, and because also $-1 \le E_i(i\mathbb{E}Z_i, r_i) \le 0$ from (A.14),

$$\begin{aligned} \mathbb{P}[Z_i = 1] &\ge \mathbb{E}Z_i\big(1 + \varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big)\Big(1 - \frac{(r_i - 1)\mathbb{E}Z_i}{r_i}\big(1 + E_i(i\mathbb{E}Z_i, r_i)\big)\Big) \\ &\ge \mathbb{E}Z_i\big(1 + \varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big)\big(1 - \mathbb{E}Z_i\big) \\ &\ge \mathbb{E}Z_i - \mathbb{E}Z_i\Big(\big|\varepsilon_{i1}(i\mathbb{E}Z_i, r_i)\big| + \mathbb{E}Z_i\Big)\,. \end{aligned}$$

Now (A.9) follows. Inequalities (A.10) and (A.12) are then immediate consequences of $2\mathbb{P}[Z_i \ge 2] \le \mathbb{E}Z_i - \mathbb{P}[Z_i = 1]$ and $k\mathbb{P}[Z_i = k] \le \mathbb{E}Z_i - \mathbb{P}[Z_i = 1]$, respectively, whereas (A.11) follows from (A.9) and (A.10), because of

$$|1 - \mathbb{P}[Z_i = 0] - \mathbb{E}Z_i| \le |\mathbb{P}[Z_i = 1] - \mathbb{E}Z_i| + \mathbb{P}[Z_i \ge 2]\,.$$

For the bound on the total variation distance, we consider a random variable $U_i \sim \mathrm{Be}(1 \wedge \mathbb{E}Z_i)$. It follows that

$$
\begin{aligned}
d_{\mathrm{TV}}\big(\mathscr{L}(Z_i), \mathrm{Be}(1 \wedge \mathbb{E}Z_i)\big) &\leq \sum_{k=1}^{\infty} \big|\mathbb{P}[Z_i = k] - \mathbb{P}[U_i = k]\big| \\
&\leq |\mathbb{P}[Z_i = 1] - \mathbb{E}Z_i| + \mathbb{E}Z_i - (1 \wedge \mathbb{E}Z_i) + \mathbb{P}[Z_i \geq 2]\,.
\end{aligned}
$$

Then we invoke (A.9) and (A.10) once more.

If $\mathbb{E}Z_i = 0$, then $\varepsilon_{i1}(i\mathbb{E}Z_i, r_i) = 0$ as well, by definition. In this case all the inequalities (A.9) up to (A.13) are trivially satisfied; both the left and right hand sides are equal to 0. □

A.2.2 Bounds used in combination with A-convergence

In this subsection we use the notation of Subsection 2.2.1, that is, in particular, $\tilde{x}_n(m,x)$ from (2.14).

Lemma A.14. *Let $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ and $\mathfrak{y} := \{y_i\}_{i\in\mathbb{N}}$ be a real-valued sequences and $x \in \mathbb{R}$. Let $l \in \mathbb{Z}_+$ and $m, n \in \mathbb{N}$ such that $l < n$. It follows that*

$$
\begin{aligned}
\Big|\sum_{i=l+1}^{n} (x_i - x) y_i\Big| \leq 2x'_{\mathrm{sup}} \sum_{i=l+1}^{(a+1)m} |y_i| + 2x'_{\mathrm{sup}} \sum_{j=a+1}^{b-1} \sum_{i=1}^{m} |y_{jm+i} - y_{jm}| \\
+ \tilde{x}_n(m,x) \sum_{i=(a+1)m+1}^{bm} |y_i| + 2x'_{\mathrm{sup}} \sum_{i=bm+1}^{n} |y_i|\,,
\end{aligned}
\tag{A.15}
$$

where $a := \lfloor l/m \rfloor$, $b := \lfloor n/m \rfloor$ and $x'_{\mathrm{sup}} := |x| \vee \sup_{i\in\mathbb{N}} |x_i|$. If $\mathfrak{x}$ is a non-negative sequence and $x \geq 0$, the three factors $2x'_{\mathrm{sup}}$ in (A.15) *reduce to x'_{sup}.*

Proof. If $x'_{\mathrm{sup}} = \infty$ then (A.15) is clearly satisfied. Therefore, assume that $x'_{\mathrm{sup}} < \infty$. It follows that

$$
\begin{aligned}
&\sum_{i=l+1}^{n} (x_i - x) y_i \\
&\quad = \sum_{i=l+1}^{(a+1)m} (x_i - x) y_i + \sum_{i=(a+1)m+1}^{bm} (x_i - x) y_i + \sum_{i=bm+1}^{n} (x_i - x) y_i \\
&\quad = \sum_{i=l+1}^{(a+1)m} (x_i - x) y_i + \sum_{j=a+1}^{b-1} \sum_{i=1}^{m} (x_{jm+i} - x) y_{jm+i} + \sum_{i=bm+1}^{n} (x_i - x) y_i
\end{aligned}
$$

$$= \sum_{i=l+1}^{(a+1)m} (x_i - x)y_i + \sum_{j=a+1}^{b-1} \sum_{i=1}^{m} \Big(x_{jm+i} - \frac{1}{m} \sum_{k=1}^{m} x_{jm+k} \Big) (y_{jm+i} - y_{jm})$$
$$+ \sum_{j=a+1}^{b-1} \sum_{i=1}^{m} \Big(\frac{1}{m} \sum_{k=1}^{m} x_{jm+k} - x \Big) y_{jm+i} + \sum_{i=bm+1}^{n} (x_i - x)y_i \,,$$

noting that

$$\sum_{j=a+1}^{b-1} \sum_{i=1}^{m} \Big(x_{jm+i} - \frac{1}{m} \sum_{k=1}^{m} x_{jm+k} \Big) y_{jm} = 0 \qquad \text{for all } a < j < b.$$

We conclude (A.15). If $\mathfrak{x}$ is a non-negative and $x \geq 0$, then we have $|x_i - x| \leq x'_{\mathrm{sup}}$ instead of $|x_i - x| \leq 2x'_{\mathrm{sup}}$, for all $i \in \mathbb{N}$. □

For a real-valued sequence $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ and an $x \in \mathbb{R}$ we set

$$\hat{x}_n(l,x) := \sum_{i=l+1}^{n} \frac{x_i - x}{i} \qquad \text{for all } l \in \mathbb{Z}_+ \text{ and } n \in \mathbb{N}. \tag{A.16}$$

Lemma A.15. *Let $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ be a real-valued sequence, $x \in \mathbb{R}$, and $x'_{\mathrm{sup}} := |x| \vee \sup_{i\in\mathbb{N}}|x_i|$. Let $l \in \mathbb{Z}_+$ and $m, n \in \mathbb{N}$ such that $l < n$.*

(i) If $l = 0$ it follows that

$$\big|\hat{x}_n(l,x)\big| \leq \Big(4 + \frac{\pi^2}{3}\Big)x'_{\mathrm{sup}} + 4x'_{\mathrm{sup}} \log m + \tilde{x}_n(m,x)(1 + \log n)\,. \tag{A.17}$$

(ii) If $1 \leq l < m$ it follows that

$$\big|\hat{x}_n(l,x)\big| \leq \Big(4 + \frac{\pi^2}{3}\Big)x'_{\mathrm{sup}} + 4x'_{\mathrm{sup}} \log m + \tilde{x}_n(m,x)\log(n/l)\,. \tag{A.18}$$

(iii) If $m \leq l < n$ it follows that

$$\big|\hat{x}_n(l,x)\big| \leq 4x'_{\mathrm{sup}} \log(1 + 1/\lfloor l/m \rfloor) + 2x'_{\mathrm{sup}} \sum_{j=\lfloor l/m \rfloor}^{\infty} \frac{1}{j^2} + \tilde{x}_n(m,x)\log(n/l)\,. \tag{A.19}$$

Proof. If $x'_{\mathrm{sup}} = \infty$ then (i), (ii) and (iii) are clearly satisfied. Therefore, assume that $x'_{\mathrm{sup}} < \infty$. We apply Lemma A.14 with $y_i := 1/i$, $i \in \mathbb{N}$. Recalling that

$a = \lfloor l/m \rfloor$ and $b = \lfloor n/m \rfloor$, we conclude that

$$\begin{aligned}
\left|\hat{x}_n(l,x)\right| &\le 2x'_{\sup} \sum_{i=am+1}^{(a+1)m} \frac{1}{i} + 2x'_{\sup} \sum_{j=a+1}^{b-1} \sum_{i=1}^{m} \frac{i}{jm(jm+i)} \\
&\quad + \tilde{x}_n(m,x) \sum_{j=a+1}^{b-1} \sum_{i=1}^{m} \frac{1}{jm+i} + 2x'_{\sup} \sum_{i=bm+1}^{(b+1)m} \frac{1}{i} \qquad \text{(A.20)} \\
&\le 4x'_{\sup} \sum_{i=am+1}^{(a+1)m} \frac{1}{i} + 2x'_{\sup} \sum_{j=a+1}^{b-1} \frac{1}{j^2} + \tilde{x}_n(m,x) \sum_{i=l+1}^{n} \frac{1}{i} \,.
\end{aligned}$$

In (i) we have $l = 0$, and thus $a = 0$. Invoking (A.2) and (A.4), we obtain

$$\left|\hat{x}_n(l,x)\right| \le 4x'_{\sup}(1 + \log m) + \frac{\pi^2 x'_{\sup}}{3} + \tilde{x}_n(m,x)(1 + \log n)\,.$$

In (ii) we have $1 \le l < m$. Here, $a = 0$ as well. It follows with (A.2), (A.3) and (A.4) that

$$\left|\hat{x}_n(l,x)\right| \le \Big(4 + \frac{\pi^2}{3}\Big) x'_{\sup} + 4x'_{\sup} \log m + \tilde{x}_n(m,x) \log(n/l)\,.$$

In (iii) we assume that $m \le l < n$, so that $a \ge 1$. Inequality (A.3) then implies that

$$\left|\hat{x}_n(l,x)\right| \le 4x'_{\sup} \log(1 + 1/a) + 2x'_{\sup} \sum_{j=a}^{\infty} \frac{1}{j^2} + \tilde{x}_n(m,x) \log(n/l)\,. \qquad \square$$

If we restrict ourselves to a non-negative sequence $\mathfrak{x}$ and $x \ge 0$, and if we assume that $m \le n$, the bounds of Lemma A.15 can be improved slightly.

Lemma A.16. *Let* $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ *be a non-negative real-valued sequence,* $x \ge 0$, *and* $x'_{\sup} := x \vee \sup_{i\in\mathbb{N}} x_i$. *Let* $l \in \mathbb{Z}_+$ *and* $m, n \in \mathbb{N}$ *such that* $l < n$ *and* $m \le n$. *Also recall the definition of* $\hat{x}_n(l,x)$ *in* (A.16).

(i) If $l = 0$ *it follows that*

$$\left|\hat{x}_n(l,x)\right| \le \Big(2 + \frac{\pi^2}{6} + \log 2\Big) x'_{\sup} + x'_{\sup} \log m + \tilde{x}_n(m,x) \log n\,. \qquad \text{(A.21)}$$

(ii) If $1 \le l < m$ *it follows that*

$$\left|\hat{x}_n(l,x)\right| \le \Big(1 + \frac{\pi^2}{6} + \log 2\Big) x'_{\sup} + x'_{\sup} \log m + \tilde{x}_n(m,x) \log(n/l)\,. \qquad \text{(A.22)}$$

(iii) If $m \le l < n$ it follows that

$$\left|\hat{x}_n(l,x)\right| \le 2x'_{\sup}\log(1+1/\lfloor l/m\rfloor)+x'_{\sup}\sum_{j=\lfloor l/m\rfloor}^{\infty}\frac{1}{j^2}+\tilde{x}_n(m,x)\log(n/l)\,. \quad \text{(A.23)}$$

Proof. We adopt the notation from the proof of Lemma A.15. Because $\mathfrak{x}$ and x are non-negative, we obtain, in place of (A.20), the inequality

$$\left|\hat{x}_n(l,x)\right| \le x'_{\sup}\sum_{i=am+1}^{(a+1)m}\frac{1}{i}+x'_{\sup}\sum_{j=a+1}^{b-1}\frac{1}{j^2}+\tilde{x}_n(m,x)\sum_{i=l+1}^{n}\frac{1}{i}+x'_{\sup}\sum_{i=bm+1}^{(b+1)m}\frac{1}{i}\,.$$

Since $m \le n$, we have $b \ge 1$. Thus, the last summand in the previous inequality can be bounded by $x'_{\sup}\log 2$, using (A.3). Arguing much as in the proof of Lemma A.15, we now obtain under condition (i)

$$\left|\hat{x}_n(l,x)\right| \le \Big(1+\frac{\pi^2}{6}\Big)x'_{\sup}+x'_{\sup}\log m+\tilde{x}_n(m,x)(1+\log n)+x'_{\sup}\log 2\,,$$

under condition (ii)

$$\left|\hat{x}_n(l,x)\right| \le \Big(1+\frac{\pi^2}{6}\Big)x'_{\sup}+x'_{\sup}\log m+\tilde{x}_n(m,x)\log(n/l)+x'_{\sup}\log 2\,,$$

and under condition (iii)

$$\left|\hat{x}_n(l,x)\right| \le 2x'_{\sup}\log(1+1/a)+x'_{\sup}\sum_{j=a}^{\infty}\frac{1}{j^2}+\tilde{x}_n(m,x)\log(n/l)\,.$$

This proves the lemma. $\square$

The following lemma plays an important role in the proof of the Dickman approximation theorems.

Lemma A.17. *Let $\mathfrak{x} := \{x_i\}_{i\in\mathbb{N}}$ be a non-negative sequence, $x \ge 0$, and $x'_{\sup} := x \vee \sup_{i\in\mathbb{N}} x_i$. Let X be a $\mathbb{Z}_+$-valued random variable. Let $l \in \mathbb{Z}_+$ and $m,n \in \mathbb{N}$ such that $l < n$.*

(i) For any bounded function $g : \mathbb{Z}_+ \to \mathbb{R}$ we have

$$\begin{aligned}&\Big|\sum_{i=l+1}^{n}(x_i-x)\mathbb{E}g(X+i)\Big|\\ &\quad\le \|g\|\Big(2x'_{\sup}mn\,d_{\mathrm{TV}}\big(\mathscr{L}(X),\mathscr{L}(X+1)\big)+2x'_{\sup}m+\tilde{x}_n(m,x)n\Big)\,. \quad \text{(A.24)}\end{aligned}$$

(ii) For every $k \in \mathbb{N}$ it follows that

$$\begin{aligned}\Big|\sum_{i=l+1}^{n}(x_i - x)\mathbb{P}[X+i=k]\Big| &\le 2x'_{\sup} m^2 d_{\mathrm{TV}}\big(\mathscr{L}(X), \mathscr{L}(X+1)\big)\\ &\quad + x'_{\sup}\mathbb{P}\big[k-n \le X < k - \lfloor n/m \rfloor m\big]\\ &\quad + x'_{\sup}\mathbb{P}\big[k - (\lfloor l/m \rfloor + 1)m \le X < k - l\big]\\ &\quad + \tilde{x}_n(m,x)\\ &\le 2x'_{\sup} m^2 d_{\mathrm{TV}}\big(\mathscr{L}(X), \mathscr{L}(X+1)\big)\\ &\quad + 2x'_{\sup} m \sup\nolimits_{j\in\mathbb{Z}_+} \mathbb{P}[X=j] + \tilde{x}_n(m,x)\,.\end{aligned} \tag{A.25}$$

Proof. We may assume that $x'_{\sup} < \infty$; otherwise (A.24) and (A.25) hold trivially.

(i) We apply Lemma A.14 with $y_i := \mathbb{E}g(X+i)$ for all $i \in \mathbb{N}$. In this situation (A.15) yields

$$\begin{aligned}&\Big|\sum_{i=l+1}^{n}(x_i - x)\mathbb{E}g(X+i)\Big|\\ &\quad\le x'_{\sup}\|g\|m + x'_{\sup}\sum_{j=1}^{b-1}\sum_{i=1}^{m}\big|\mathbb{E}\{g(X+jm+i) - g(X+jm)\}\big|\\ &\qquad + \tilde{x}_n(m,x)\|g\|n + x'_{\sup}\|g\|m\,.\end{aligned} \tag{A.26}$$

Since for all $1 \le i \le m$

$$\big|\mathbb{E}\{g(X+jm+i) - g(X+jm)\}\big| \le 2\|g\| i d_{\mathrm{TV}}\big(\mathscr{L}(X), \mathscr{L}(X+1)\big)\,,$$

we conclude (A.24) from (A.26), recalling from Lemma A.14 that $b \le n/m$.

(ii) For any fixed $k \in \mathbb{N}$ we define $y_i := \mathbb{P}[X+i=k]$ for all $i \in \mathbb{N}$. Then (A.15) leads to

$$\begin{aligned}&\Big|\sum_{i=l+1}^{n}(x_i - x)\mathbb{P}[X+i=k]\Big|\\ &\quad\le x'_{\sup}\mathbb{P}\big[k-(a+1)m \le X < k-l\big]\\ &\qquad + x'_{\sup}\sum_{j=1}^{b-1}\sum_{i=1}^{m}\big|\mathbb{P}[X+jm+i=k] - \mathbb{P}[X+jm=k]\big|\\ &\qquad + \tilde{x}_n(m,x) + x'_{\sup}\mathbb{P}\big[k-n \le X < k-bm\big]\,,\end{aligned} \tag{A.27}$$

with $a = \lfloor l/m \rfloor$ and $b = \lfloor n/m \rfloor$. We have

$$\sum_{j=1}^{b-1} \bigl|\mathbb{P}[X + jm + i = k] - \mathbb{P}[X + jm = k]\bigr| \le 2i d_{\mathrm{TV}}\bigl(\mathscr{L}(X), \mathscr{L}(X+1)\bigr)$$

for all $1 \le i \le m$. Now (A.25) follows from (A.27). $\square$

A.3 Open problems

In Sections 3.2 and 3.3

It seems possible that the global and local Dickman approximation theorems, Theorem 3.14 and Theorem 3.21, hold true if the $\mathsf{A}(\theta)$-convergence of $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ in $\mathsf{SUQLC}(\theta, \mathfrak{r}, \mathfrak{b})$ is replaced by Cesàro convergence, that is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} i\mathbb{E}Z_i = \theta\,,$$

and the assumption that $\sup_{i\in\mathbb{N}} i\mathbb{E}Z_i < \infty$ (in fact, if we are only interested in distributional approximation, these relaxed assumption suffice, see Lemma 3.17).

This consideration is inspired by the following. Let $\{Z_i^*\}_{i\in\mathbb{N}}$ be a sequence of independent Poisson random variables such that $\sup_{i\in\mathbb{N}} i\mathbb{E}Z_i^* < \infty$. Arratia et al. (1995, Lemma 7) show that $n^{-1}T_{0,n}^*$, where $T_{0,n}^* := \sum_{i=1}^n iZ_i^*$, converges in distribution to the generalized Dickman distribution $\mathrm{GD}(\theta)$, if and only if $\{i\mathbb{E}Z_i^*\}_{i\in\mathbb{N}}$ converges to θ in the sense of Cesàro.

Now let $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ be a sequence of independent $\mathbb{Z}_+$-valued random variables with $\mathbb{E}Z_i = \mathbb{E}Z_i^*$ for all $i \in \mathbb{N}$. Lemmas 3.6 and 3.7 show that the deviation of the distribution of $T_{0,n} := \sum_{i=1}^n iZ_i$ from the compound Poisson distribution $\mathscr{L}(T_{0,n}^*)$ depends on $K_{0,n}^{(1)}(\mathfrak{r}, g)$, as defined in (3.10), which is controlled through condition $\mathsf{UC}(\mathfrak{r})$. This uniformity condition in turn does not contain any restrictions on how the sequence $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ should behave. In particular, it does not incorporate the $\mathsf{A}(\theta)$ convergence of $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$, which is controlled separately through $K_{0,n}^{(2)}(\mathfrak{r}, g)$, defined in (3.11).

Therefore, it may be possible to establish the Theorem 3.14 and Theorem 3.21 by first comparing $\mathscr{L}(T_{0,n})$ with $\mathscr{L}(T_{0,n}^*)$ and then by a comparison of $\mathscr{L}(n^{-1}T_{0,n}^*)$ and $\mathrm{GD}(\theta)$ under the Cesàro convergence assumption. We state the following two conjectures:

Conjecture A.18. *Let $\theta > 0$. Let $\mathfrak{r}$ be a sequence of positive integers and $\mathfrak{b} := \{b_n\}_{n\in\mathbb{N}}$ a sequence of non-negative integers. Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence* *of $\mathbb{Z}_+$-valued random variables that satisfies conditions* $\mathsf{UC}(\mathfrak{r})$ *and* $\mathsf{SC}(\mathfrak{b})$*. If* $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ *is bounded and converges to* θ *in the sense of Cesàro, then it follows that*

$$\lim_{n\to\infty} d_{\mathrm{W}}\big(\mathscr{L}(n^{-1}T_{a_n,n}), \mathrm{GD}(\theta)\big) = 0$$

for every non-negative integer sequence $\{a_n\}_{n\in\mathbb{N}}$ *that satisfies* $a_n \le b_n$ *for all* $n \in \mathbb{N}$*.*

Conjecture A.19. *Let* $\theta > 0$*. Let* $\mathfrak{r}$ *be a sequence of positive integers and* $\mathfrak{b} := \{b_n\}_{n\in\mathbb{N}}$ *a sequence of non-negative integers such that* $b_n = \mathrm{o}(n)$*. Let* $\{Z_i\}_{i\in\mathbb{N}}$ *be a sequence of* $\mathbb{Z}_+$*-valued random variables that satisfies conditions* $\mathsf{UC}(\mathfrak{r})$ *and* $\mathsf{SC}(\mathfrak{b})$*, and such that the sequence* $\{i\mathbb{E}Z_i\}_{i\in\mathbb{N}}$ *is bounded and converges to* θ *in the sense of Cesàro. If* $\{k_n\}_{n\in\mathbb{N}}$ *is a sequence of non-negative integers such that, for some* $x > 0$*,*

$$\lim_{n\to\infty} \frac{k_n}{n} = x\,,$$

then it follows that

$$\lim_{n\to\infty} n\mathbb{P}\big[T_{a_n,n} = k_n\big] = p_\theta(x)$$

for every non-negative integer sequence $\{a_n\}_{n\in\mathbb{N}}$ *that satisfies* $a_n \le b_n$ *for all* $n \in \mathbb{N}$*.*

Whereas the assumption of condition of $\mathsf{SC}(\mathfrak{b})$ in Conjecture A.18 maybe could be dropped, it is necessary in Conjecture A.19. Indeed, if for example $i\mathbb{E}Z_i = 0$ for each odd i and $i\mathbb{E}Z_i = 1$ for each even i then condition $\mathsf{SC}(\mathfrak{b})$ is never satisfied, and $n\mathbb{P}[T_{0,n} = n]$ does not converge.

In Subsection 4.3.2

Let $\mathcal{A} := (A, \circ, e, P, \|\cdot\|)$ be an additive arithmetic semigroup with counting function $a(n)$ of A and $p(n)$ of P. In Theorem 4.33 we show that a necessary condition on $a(n)$ that $\mathcal{A}$ is (θ, x)-quasi-logarithmic for some $\theta > 0$ and $0 < x < 1$ is

$$a(n) \sim c x^{-n} n^{\theta-1} \ell(n)\,,$$

for a constant $c > 0$ and a slowly varying function $\ell(n)$, which can, depending on the AAS $\mathcal{A}$, exhibit every possible asymptotic growth behaviour a slowly varying function allows (cf. Remark 4.35).

In Theorem 4.32 we show that $\mathcal{A}$ is (θ, x)-quasi-logarithmic if the counting function $a(n)$ has a special form which, in particular, entails that

$$a(n) \sim c' x^{-n} n^{\theta-1}\,,$$

for some $c' > 0$. This can be seen as a representation with a convergent slowly varying function $\ell(n)$.

Question A.20. *What is a necessary and sufficient condition on $a(n)$ for the AAS $\mathcal{A}$ to be quasi-logarithmic?*

In view of the results of Zhang (1996b) and Wehmeier (2004) (cf. Subsubsection 5.2.4.1) one can also pose the following simpler question.

Question A.21. *How slowly can a sequence r_n converge to 0 in order that an AAS $\mathcal{A}$ with*

$$a(n) = cx^{-n} + \mathrm{O}\big(x^{-n} r_n\big)$$

still is quasi-logarithmic? Recall that if $r_n = \mathrm{O}(n^{-\delta})$, $\delta > 2$, Theorem 4.32 tells us that $\mathcal{A}$ is quasi-logarithmic.

In Subsection 5.2.2

The convergence rates of the total variation distance examined in Corollary 5.21 do not seem to be optimal, as can be seen from the discussion in Example 5.23. In this example, hybrid ESF-structures are considered, and it can be seen in this case that the convergence rate depends only on α_2 and not on α_1 and α_3 (recall (5.35), (5.36) and (5.37) for their definitions). The parameter α_2 results from the coupling developed in Chapter 6. The question is, whether the coupling can be further improved to give, at least in the case of hybrid ESF-structures, better convergence rates than the one in Corollary 5.21.

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